Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **17**, No. 3, pp. 989-1013

DOI: 10.4208/aamm.OA-2023-0288
June 2025

Front Tracking and Parameter Identification for a Conservation Law with a Space-Dependent Coefficient Modeling Granular Segregation

Raimund Bürger[1,*], Yessennia Martínez[1] and Luis M. Villada[2]

[1] *CI²MA and Departamento de Ingeniería Matemática, Facultad de Ciencias Físicas y Matemáticas, Universidad de Concepción, Casilla 160-C, Concepción, Chile*
[2] *GIMNAP-Departamento de Matemática, Universidad del Bío-Bío, Casilla 5-C, Concepción, Chile and CI²MA, Universidad de Concepción, Casilla 160-C, Chile*

Received 5 November 2023; Accepted (in revised version) 26 May 2024

Abstract. A well-known experimental setup for the study of segregation by size in a dry granular medium consists of two layers of spheres composed of large and small rigid spheres. These layers are contained within an annular region of concentric cylinders covered above and below by plates. One of the cylinders is rotated and thereby applies shear to the granular mixture. The spheres will then mix and the large ones rise while the small ones settle in vertical direction. This phenomenon can be modelled by a conservation law whose flux involves a piecewise constant or smooth coefficient [L. May, M. Shearer, and K. Daniels, J. Nonlin. Sci., 20 (2010), pp. 689–707] that describes dependence of the shear rate on depth. This model is solved by the hyperfast front tracking method adapted to a conservation law with discontinuous flux. In this way the coefficient can efficiently be identified from experimental observations. Numerical examples are presented.

AMS subject classifications: 76T25, 35L65, 35F25
Key words: Granular media, segregation, conservation law, discontinuous flux, front tracking method, parameter identification.

1 Introduction

1.1 Scope

The well-known front tracking (FT) method was originally devised for the approximate solution of initial value problems of scalar conservation laws of the type

$$\partial_t u + \partial_x g(u) = 0, \quad x \in \mathbb{R}, \quad t > 0; \quad u(x,0) = u_0(x), \quad x \in \mathbb{R}, \tag{1.1}$$

*Corresponding author.
Emails: rburger@ing-mat.udec.cl (R. Bürger), yesmartinez@udec.cl (Y. Martínez), lvillada@ubiobio.cl (L. Villada)

where $u=u(x,t)$ is the sought unknown, t is time, x is the spatial coordinate, $g=g(u)$ is the given flux function that we may assume to be Lipschitz continuous, and $u_0\in(BV\cap L^1)(\mathbb{R})$ is an initial function. The method essentially goes back to the polygonal construction by Dafermos [14], while Holden et al. [18] proved that the method is well defined and suitable for the approximation of entropy solutions of (1.1) with a general nonlinear flux g. The FT method is based on replacing the flux g by a piecewise linear, continuous (that is, polygonal) function g^δ, where $\delta>0$ is a parameter that controls the accuracy of polygonal approximation (such that $g^\delta\to g$ as $\delta\to 0$ in an appropriate sense), and replacing u_0 by a piecewise constant function u_0^δ that takes values in the breakpoints of g^δ only. The resulting initial value problem

$$\partial_t u^\delta+\partial_x g^\delta(u^\delta)=0,\quad x\in\mathbb{R},\quad t>0;\quad u^\delta(x,0)=u_0^\delta(x),\quad x\in\mathbb{R},\tag{1.2}$$

is then solved exactly for fixed $\delta>0$. The solution of (1.2) starts by solving the initial neighboring Riemann problems posed by the jumps of u_0^δ; here we recall that a Riemann problem is an initial value problem (1.1) with

$$u_0(x)=\begin{cases}u_{\mathrm{L}} & \text{for } x<0,\\ u_{\mathrm{R}} & \text{for } x>0,\end{cases}$$

with given constants u_{L} and u_{R}. One then alternates between tracking fronts, that is traveling discontinuities and detecting their interactions, and solving the new Riemann problems defined by these interactions. By construction the solution $u^\delta=u^\delta(x,t)$ is constant between the traveling fronts and takes values within the breakpoints of g^δ only. The solution of the individual Riemann problems for (1.2) is based on the construction of lower convex and upper concave envelopes of g^δ and avoids the construction of rarefaction waves for a continuously varying function g.

All these properties of the FT method for (1.1), along with the convergence of u^δ in L^1 to the unique entropy solution of (1.1) as $\delta\to 0$, are well known and detailed in the monograph by Holden and Risebro [20]. In particular, the method is known to be hyperfast, which means that it delivers a solution of (1.2) up to infinite time within finite computational time [20]. On the other hand, the method can be applied to handle conservation laws with a flux that is discontinuous in spatial position

$$\partial_t u+\partial_x g(x,u)=0,\tag{1.3}$$

where a typical situation is

$$g(x,u)=k(x)f(u),\quad k(x)=\begin{cases}k_{\mathrm{L}} & \text{for } x\le 0,\\ k_{\mathrm{R}} & \text{for } x>0.\end{cases}\tag{1.4}$$

A more general case would be

$$g(x,u)=\begin{cases}f_{\mathrm{L}}(u) & \text{for } x\le 0,\\ f_{\mathrm{R}}(u) & \text{for } x>0.\end{cases}\tag{1.5}$$