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Thirteen-Velocity Three-Dimensional Multiple-Relaxation-Time Lattice Boltzmann Model for Incompressible Navier-Stokes Equations

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Abstract. This paper presents a three-dimensional thirteen-velocity (D3Q13) multiple-relaxation-time (MRT) lattice Boltzmann (LB) model for incompressible flows. The model can eliminate the second kind of compressibility error existing in previous D3Q13 MRT LB models. We simulate the three-dimensional steady Poiseuille flow, unsteady pulsatile flow, and lid-driven cavity flow to verify the validity of the current model. The excellent agreement between the numerical result and the analytical solution or the previous numerical result demonstrates the model's effectiveness.

AMS subject classifications: 76M25, 76D05

Key words: Lattice Boltzmann method, multiple relaxation time, incompressible Navier-Stokes equations, compressibility error.

1 Introduction

In the past decades, researchers have accepted the LB method as an effective numerical method for solving incompressible Navier-Stokes (N-S) equations [1–3]. Compressibility error is considered an essential issue in solving incompressible N-S equations by the LB model [4–8]. However, few papers have been published to elucidate the compressibility

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error occurring in the LB method [9–12]. Generally, the compressibility error can be attributed to the errors of two kinds. The first is the error caused by the non-zero Mach number. For example, the accuracy of the macroscopic momentum equations of most LB models is of order $\mathcal{O}(\delta t^2+\delta t M^2)$, where δt denotes the time step and M denotes the Mach number. If the Mach number is not zero, the error such as $\mathcal{O}(\delta t M^2)$ always exists. Essentially, this error is caused by the compressibility of the flow, so it can be named the compressibility error of the first kind. The deviation of the macroscopic equations of the LB models from the incompressible N-S equations causes the second kind of compressibility error. Under the low Mach number assumption, we expect an LB model to recover to the incompressible N-S equations because the low Mach number assumption is equivalent to the incompressible limit. Unfortunately, most LB models can only recover to the compressible N-S equations under the low Mach number assumption. When we use these LB models to simulate incompressible flows, the LB solution may deviate from the solution of the incompressible N-S equations. This deviation is called the compressibility error of the second kind.

In the past, various efforts have focused on the second kind of compressibility error [4–8]. For example, Guo et al. [4] proposed a two-dimensional nine-velocity (D2Q9) lattice Bhatnagar-Gross-Krook (LBGK) model, which can eliminate the compressibility error of the second kind in the existing two-dimensional (2D) LBGK models [12, 13]. He et al. [5] presented a unified DdQq incompressible LBGK model for 2D and three-dimensional (3D) incompressible flows, which can eliminate the compressibility error of the second kind in the existing 3D LBGK models. Based on the equilibrium distribution functions of Guo et al. D2Q9 LBGK model, Du et al. [6] and Du [7] developed new versions of D2Q9 and D3Q19 MRT LB models for 2D and 3D incompressible flows, respectively. Zhang et al. [8] proposed an LBGK D2Q9 model for incompressible axisymmetric flows, which can eliminate the compressibility error of the second kind in the existing LB models for axisymmetric flows.

In the LB method, the commonly used 2D and 3D models are D2Q9, D3Q13, D3Q15, D3Q19 and D3Q27 models [13–16]. Generally, the MRT LB model is much more stable than the LBGK model. However, the computational speed of the MRT LB model in terms of the number of nodes updated per second is about 15% slower than that of the LBGK model [15]. To realize a computationally more efficient two-dimensional MRT LB model for incompressible flows, Du et al. proposed a D2Q8 MRT LB model [17]. Enlightened by Du et al.'s work, Zhang et al. proposed D3Q14 and D3Q18 MRT LB models for three-dimensional incompressible flows [18].

To facilitate the discussion, we refer to all the above-discussed LB models as the scalar LB models [19]. In the 3D space, a scalar LB model usually needs at least 13 discrete velocity directions to solve the incompressible N-S equations. Following the developments of the D2Q8, D3Q14, and D3Q18 MRT LB models, an intuitive question is if a D3Q12 MRT LB model can be constructed similarly. This realization could be instrumental because a D3Q12 MRT LB model is, in principle, the computationally most efficient 3D MRT LB model. However, constructing such a model is challenging because the method adopted