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Discontinuous Galerkin Methods for Auto-Convolution Volterra Integral Equations

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Abstract. The discontinuous Galerkin (DG) method is employed to solve the auto-convolution Volterra integral equations (AVIEs). The solvability of the DG method is discussed, then it is proved that the quadrature DG (QDG) method obtained from the DG method by approximating the inner products by suitable numerical quadrature formulas, is equivalent to the piecewise discontinuous polynomial collocation method. The uniform boundedness of the DG solution is provided by defining a discrete weighted exponential norm, and the optimal global convergence order of the DG solution is obtained. In order to improve the numerical accuracy, the iterated DG method is introduced. By virtue of a projection operator, the optimal $m+1$ superconvergence order of the iterated DG solution is gained, as well as $2m$ local superconvergence order at mesh points. It is noting that both the global and local superconvergence are obtained under the same regularity assumption as that for the convergence, other than the collocation method, one has to improve the regularity of the exact solution to obtain the superconvergence of the iterated collocation method. Some numerical experiments are given to illustrate the theoretical results.

AMS subject classifications: 65R20

Key words: Auto-convolution, Volterra integral equations, discontinuous Galerkin method, convergence, superconvergence.

1 Introduction

Auto-convolution Volterra integral equation (AVIE) has rich applications in the theory of viscoelasticity (see [4,6,7,10,19,21] and the references cited therein), the nonlinear optics in the context of ultrashort laser pulse characterization (see [9]) and the computation of certain special functions (see [20]). For example, one may propose nonlinear stress-strain relations of the form

$$f(t)=Eu(t)+\int_0^t k_1(t,s)u(s)\mathrm{d}s+\int_0^t k_0(s)k_2(t-s)F_1(u(t-s))F_2(u(s))\mathrm{d}s$$

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in the theory of viscoclasticity, where $t\in I:=[0,T]$, f is the stress, u is the strain and $E>0$ is a constant. The determination of u by given functions f, k_0, k_1, k_2, F_1, F_2 then is a more general AVIE. These applications appeal lots of researchers focusing on the AVIE (see, for example, [5,12–15,19,21,25]).

In this paper, we mainly deal with the special case of the above equation

$$u(t)=g(t)+\int_0^t K(t,s)u(t-s)u(s)\mathrm{d}s, \quad t\in I, \tag{1.1}$$

where the function $g\in C(I)$ and the kernel function $K\in C(D)$ $(D:=\{(t,s):0\le s\le t\le T\})$ are given.

In [2, 5, 20], the solvability of the special case of AVIE (1.1) with non-zero constant value kernel function is studied extensively by the Laplace transform. For example, the first-order Bessel function $J_1(t)$ is the solution of AVIE (1.1) with $g(t)=\frac{1}{2}\sin(t)$ and $K(t,s)\equiv\frac{1}{2}$. In [25], the existence, uniqueness and regularity of solution of AVIE (1.1) with general kernel function are investigated in detail, and it shows that if $g\in C^d(I)$ and $K\in C^d(D)$ with the integer $d\ge 0$, then (1.1) has a unique solution $u\in C^d(I)$.

It is well known that for Volterra integral equations (VIEs), the natural approximated space is the discontinuous piecewise polynomial space (see [1, Section 2.2])

$$S_{m-1}^{(-1)}(I_h):=\{v:v|_{\sigma_n}\in\pi_{m-1}\ (0\le n\le N-1)\},$$

where $I_h:=\{t_n=nh:n=0,1,\cdots,N\ (t_N:=T)\}$ denotes a uniform mesh on I, $\sigma_n:=(t_n,t_{n+1}]$, and π_{m-1} denotes the set of real polynomials of degree not exceeding $m-1$ with the integer $m\ge 1$. In [25], it is shown that the global convergence order of the collocation solution in $S_{m-1}^{(-1)}(I_h)$ is m for $g\in C^m(I)$ and $K\in C^m(D)$. Further, for the iterated collocation solution, if the collocation parameters satisfy some orthogonal conditions, the $m+1$ global superconvergence order is obtained by improving the regularity to $g\in C^{m+1}(I)$ and $K\in C^{m+1}(D)$; and at mesh points, the $2m$ highest attainable local superconvergence order can be reached under the regularity $g\in C^{2m}(I)$ and $K\in C^{2m}(D)$.

In this paper, we use discontinuous Galerkin (DG) method in the same piecewise polynomial space to solve the AVIE (1.1). There is currently great interest in using DG method to solve VIEs and differential equations (see, for example, [8,11,16,17,22–24]). Similarly to [16,17], we will show that for AVIE (1.1), the quadrature DG (QDG) method obtained from the DG method by approximating the inner products by suitable numerical quadrature formulas, is equivalent to the piecewise discontinuous polynomial collocation method. It is proved that the DG solution has the same global convergence order as the collocation solution in [25] under the same regularity assumption. In addition, in order to improve the numerical accuracy, the iterated DG method is introduced similar to the iterated collocation method in [25], and the $m+1$ global superconvergence order and the $2m$ local superconvergence order at mesh points for the iterated DG solution are obtained. However, unlike the iterated collocation method, we find that it is not necessary