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Local Discontinuous Galerkin Method for the Backward Feynman-Kac Equation

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Abstract. Anomalous diffusions are ubiquitous in nature, whose functional distributions are governed by the backward Feynman-Kac equation. In this paper, the local discontinuous Galerkin (LDG) method is used to solve the 2D backward Feynman-Kac equation in a rectangular domain. The spatial semi-discrete LDG scheme of the equivalent form (obtained by Laplace transform) of the original equation is established. After discussing the properties of the fractional substantial calculus, the stability and optimal convergence rates of the semi-discrete scheme are proved by choosing an appropriate generalized numerical flux. The $L1$ scheme on the graded meshes is used to deal with the weak singularity of the solution near the initial time. Based on the theoretical results of a semi-discrete scheme, we investigate the stability and convergence of the fully discrete scheme, which shows the optimal convergence rates. Numerical experiments are carried out to show the efficiency and accuracy of the proposed scheme. In addition, we also verify the effect of the central numerical flux on the convergence rates and the condition number of the coefficient matrix.

AMS subject classifications: 65D15, 35R11

Key words: Backward Feynman-Kac equation, fractional substantial calculus, LDG method, generalized numerical flux, graded meshes, $L1$ scheme.

1 Introduction

The origin of Feynman-Kac transform can be traced back to Richard Feynman's research on "path integrals" in the 1940s. Later, Mark Kac realized that the solution of Schröndinger equation (heat equation with external potential term) describing the functional distribution of diffusion motion can be obtained by this transformation [16]. The functional of anomalous diffusion has attracted extensive attention of physicists with the

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in-depth study of non-Brownian motion and anomalous diffusion. Similar to the functional of Brownian motion, the functional of anomalous diffusion can be defined as

$$A_t = \int_0^t \kappa(Y_{E_s})ds, \tag{1.1}$$

where $\kappa(\mathbf{x})$ is a bounded function on the state space of the stochastic process Y_{E_t} with E_t being the inverse of the driftless subordinator with Lévy measure $\mu(dx) := -d\omega(x)$ that is independent of the Markov process Y_t. In fact, if Y_{E_t} is the trajectory formed by a particle undergoing Brownian (non Brownian) motion, the corresponding A_t represents the functional of Brownian (nomalous diffusion). Most of the time, the functionals have clear physical meaning. For example, we can take $\kappa(\mathbf{x}) = 1$ in a given domain, otherwise it is zero, which characterizes the time spent by a particle in that domain. The functional can be used to study the kinetics of chemical reactions that occur specifically in a particular domain. For non-uniform disordered dispersion systems, the motion of particles is non Brownian, and $\kappa(\mathbf{x})$ is taken as $\mathbf{x}$ or $\mathbf{x}^2$ [3, 10, 23].

The distribution of the functional defined by (1.1) is governed by the 2D backward equation for Feynman-Kac transform with the abstract form [5]

$$\partial_t^{\omega,\kappa(\mathbf{x})} u(t,\mathbf{x}) = \mathcal{L}u(t,\mathbf{x}) - \kappa(\mathbf{x}) I_t^{\omega,\kappa(\mathbf{x})} u(t,\mathbf{x}) \quad \text{on } (0,T] \times \Omega, \tag{1.2}$$

where $\omega(t)$ is an unbounded right continuous decreasing function on $(0,\infty)$ that is integrable on $(0,1]$ and has $\lim_{t\to+\infty}\omega(t) = 0$. $\mathcal{L}$ is the generator of the stochastic process Y_t. $\partial_t^{\omega,\kappa(x)} u(t,\mathbf{x})$ denotes the generalized time fractional derivative, defined by

$$\partial_t^{\omega,\kappa(\mathbf{x})} u(t,\mathbf{x}) = \frac{\partial}{\partial t} I_t^{\omega,\kappa(\mathbf{x})} (u(t,\mathbf{x}) - u(0,\mathbf{x})) \tag{1.3}$$

with $I_t^{\omega,\kappa(\mathbf{x})}$ being the generalized time fractional integral

$$I_t^{\omega,\kappa(\mathbf{x})} u(t,\mathbf{x}) := \int_0^t e^{-\kappa(\mathbf{x})(t-\tau)} \omega(t-\tau) u(\tau,\mathbf{x}) d\tau. \tag{1.4}$$

The boundary condition of (1.2) is periodic and the initial condition is

$$u(0,\mathbf{x}) = u_0(\mathbf{x}), \quad \mathbf{x} \in \Omega, \tag{1.5}$$

where the rectangular region $\Omega \subset \mathbb{R}^2$. The solution of (1.2) describes the probability distribution of the above functional (1.1) in frequency domain, and the Brownian (non-Brownian) functional corresponds to the integer (fractional) order Feynman-Kac equation [3, 35].

The main challenges for analyzing and solving (1.2) come from the spatiotemporal coupling and nonlocality of the generalized time fractional derivative (1.3); it is hard to get analytical solution. So, finding an effective numerical method to solve (1.2) seems to be urgent. Currently, there are many results for the numerical algorithms of fractional