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# A Nonconforming P2 and Discontinuous P1 Mixed Finite Element on Tetrahedral Grids

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**Abstract.** A nonconforming $P_2$ finite element is constructed by enriching the conforming $P_2$ finite element space with seven $P_2$ nonconforming bubble functions (out of fifteen such bubble functions on each tetrahedron). This spacial nonconforming $P_2$ finite element, combined with the discontinuous $P_1$ finite element on general tetrahedral grids, is inf-sup stable for solving the Stokes equations. Consequently such a mixed finite element method produces optimal-order convergent solutions for solving the stationary Stokes equations. Numerical tests confirm the theory.

**AMS subject classifications**: 65N15, 65N30, 76M10

**Key words**: Quadratic finite element, nonconforming finite element, mixed finite element, Stokes equations, tetrahedral grid.

## 1 Introduction

We solve the Stokes equations: Find functions $\mathbf{u}$, the fluid velocity, and $p$, the pressure in the flow, on a 3D polyhedral domain $\Omega$ such that

$$-\Delta\mathbf{u}+\nabla p=\mathbf{f} \qquad \text{in } \Omega, \tag{1.1a}$$
$$\operatorname{div}\mathbf{u}=0 \qquad \text{in } \Omega, \tag{1.1b}$$
$$\mathbf{u}=\mathbf{0} \qquad \text{on } \partial\Omega, \tag{1.1c}$$

where $\mathbf{f}\in L^2(\Omega)$. The variational form of (1.1a)–(1.1c) reads: Find $\mathbf{u}\in\mathbf{V}=H_0^1(\Omega)^3$ and $p\in P=L_0^2(\Omega)$ such that

$$\begin{aligned}
&(\nabla\mathbf{u},\nabla\mathbf{v})-(\operatorname{div}\mathbf{v},p)=(\mathbf{f},\mathbf{v}), &&\forall\mathbf{v}\in\mathbf{V},\\
&(\operatorname{div}\mathbf{u},q)=0 &&\forall q\in P.
\end{aligned}$$

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The problem is symmetric, but not positive definite. When choosing the finite element spaces for $V$ and $P$, most pairs are not stable, i.e., they fail in satisfying the inf-sup condition (3.1) below. A natural pair of finite elements is the $P_k^{\rm c}/P_{k-1}^{\rm dis}$ mixed element where $P_k^{\rm c}=\mathbf{V}_h\subset\mathbf{V}$ is the space of continuous polynomials of a tetrahedral mesh, and $P_{k-1}^{\rm dis}=P_h\subset P$ is the space of discontinuous $P_{k-1}$ polynomials on the mesh. Mostly such a method is not stable. But on Hsieh-Clough-Tocher macro triangular/tetrahedral grids [14,19,25,27], the full $P_k^{\rm c}/P_{k-1}^{\rm dis}$ space, $k\geq 2$ in 2D or $k\geq 3$ in 3D, is inf-sup stable. In all other known stable cases, the pressure space is a proper subspace of the $P_{k-1}^{\rm dis}$ space [1,2,7,8,11,15,16,18,21, 22,27–31], except when the discrete velocity is enriched by non-polynomial bubbles or subgrid-bubbles [12,13].

On the other side, it is relatively easy to find stable nonconforming $P_k^{\rm nc}/P_{k-1}^{\rm dis}$ pairs of finite elements, especially in low polynomial degree cases. Here $P_k^{\rm nc}$ means the nonconforming finite element space of polynomial degree $k$, where the piecewise polynomials are continuous up to $P_{k-1}$ order in the sense that the jump of function is orthogonal to $P_{k-1}$ polynomials on inter-element face, $\int_F [f]p_{k-1}ds=0$. Crouzeix and Raviart proposed nonconforming finite elements first [6]. In this first paper, $P_1^{\rm nc}/P_0^{\rm dis}$ elements are proved to be inf-sup stable, on triangular and tetrahedral grids. The proposed 2D $P_3^{\rm nc}/P_2^{\rm dis}$ element [6] is enriched by three $P_4^{\rm nc}$-bubbles for each component of $\mathbf{v}$. It is proved that higher-order bubbles are not needed in [5] if the triangular grid can be separated in to macro-triangles of several patterns.

Fortin and Soulie studied 2D $P_2^{\rm nc}/P_1^{\rm dis}$ mixed finite elements [10] and proved the inf-sup stability, without enriching the $P_2^{\rm nc}$ element by any higher-order bubbles. For 2D $P_k^{\rm nc}/P_{k-1}^{\rm dis}$ elements, Matthies and Tobiska enrich the velocity space by many higher-order nonconforming bubble functions so that the method is stable for all $k\geq 1$ [17]. But [3] add only one $P_k^{\rm nc}$-bubble (not high-polynomial bubbles like [17]) to each component of $P_k$ conforming finite element velocity each triangle so that the $P_k^{\rm nc}/P_{k-1}^{\rm dis}$ is stable for even polynomial degree $k\geq 2$. When $k=2$, [3] repeats the result of Fortin and Soulie [10].

In 3D, there are only two works [4] and [9] on the $P_k^{\rm nc}$ element, other than the work of first element $P_1^{\rm nc}/P_0^{\rm dis}$ in [6]. Both [4] and [9] study the 3D $P_2^{\rm nc}$ element on tetrahedral grids. But none constructs a working $P_2^{\rm nc}$ finite element space for the Stokes equations, while this work is the first one accomplishing the work. In [9], Fortin finds five $P_2^{\rm nc}$ bubbles for the scalar $P_2^{\rm c}$ finite element space on each tetrahedron. But [9] only suggests to use (some of) these bubbles to stabilize the $P_2^{\rm c}$ finite element, and does not propose a working method. In [4], Ciarlet, Dunkl and Sauter find (all) $P_k^{\rm nc}$ bubbles for the $P_k^{\rm c}$ finite element space in 3D, $k\geq 1$. When $k=2$, the bubbles of [9] and [4] are same. [4] further points out the $P_k^{\rm c}$ space and the (found) $P_k^{\rm nc}$-bubble space are not linearly independent. [4] suggests to remove all the vertex basis functions of the $P_k^{\rm c}$ finite element to obtain a unisolvent $P_k^{\rm nc}$ finite element space. But this $P_2^{\rm nc}$ finite element space is not inf-sup stable when combined with the $P_1^{\rm dis}$ pressure space for the Stokes equations.

In this work, we propose to add seven, out of fifteen, vector $P_2^{\rm nc}$-bubbles of [4,9] to the $P_2^{\rm c}$ space, on one tetrahedron. It is shown such a 3D $P_2^{\rm nc}/P_1^{\rm dis}$ finite element is inf-sup