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Mathematical Insights into Substance Addiction and Abuse Dynamics via Global Nonlocal Operators

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Abstract. This paper introduces a novel mathematical model for capturing the complex dynamics of drug abuse within populations. Departing from conventional methodologies, the model employs global derivatives to integrate non-local effects, thereby offering enhanced insight into the spread and evolution of drug abuse. The stability analysis and numerical simulations conducted in this study reveal critical thresholds and dynamic behaviors that are instrumental in understanding the persistence and potential escalation of abuse within communities. Numerical simulations also demonstrate the long-term behavior for different orders of α, and the effects of the function $g(x)$ are presented, further elucidating the intricate interplay of factors that govern the system's dynamics. These findings not only shed light on the underlying mechanisms driving the temporal and spatial patterns of drug abuse but also provide valuable guidance for designing effective intervention strategies aimed at mitigating its spread. By systematically manipulating key parameters, the model serves as a powerful tool for exploring the driving factors behind drug abuse diffusion and control. The insights gained from this research have significant implications for public health policy, offering a rigorous mathematical framework to inform targeted efforts in curbing the epidemic of drug abuse.

AMS subject classifications: 34A08

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1 Introduction

The impact of illegal drug use remains significant, resulting in the loss of numerous valuable lives and productive years. In 2022, drug-related deaths were estimated at around 11.8 million [10]. Globally, between 172 million and 424 million individuals aged 15 to 64 were estimated to have engaged in illicit drug use, [3]. Illicit drug use refers to the non-medical consumption of substances prohibited by international law, such as amphetamines, cannabis, cocaine, heroin, diploids, and MD-MA. The risks

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of premature death and health problems associated with illegal drug use depend on factors such as the amount consumed, frequency of use, and method of administration, [12, 17, 19].

Studies have shown that factors such as age, gender, socio-economic status, and mental health can all play a role in drug use and related health outcomes [1, 2, 10]. For example, younger individuals, males, and individuals with lower socio-economic status are more likely to use drugs, and are also at higher risk for adverse health outcomes. Additionally, individuals with mental health conditions, such as depression and anxiety, are more likely to use drugs, and drug use can also exacerbate existing mental health problems [12, 17].

Mathematical models have been used to better understand the dynamics of drug use and its impact on population health. These models can help researchers identify key drivers of drug use, as well as the relationships between drug use and various health outcomes, [20]. In recent years, there has also been growing interest in using fractional calculus to model drug use, as this approach allows for the exploration of complex, non-linear relationships, [9, 15, 21]. These operators provides a more advanced mathematical approach that can capture the intricate dynamics and behaviors involved since traditional calculus assumes exponential growth or decay and lacks the ability to capture long-term memory effects and non-exponential behaviors [5, 7, 15].

1.1 Motivation

Substance addiction and abuse are complex public health challenges influenced by a delicate interplay of biological, psychological, and environmental factors. These dynamics evolve over time and across different populations, making it essential to develop mathematical models that can capture both local and global influences on addiction and recovery processes. Traditional models, often based on classical or fractional derivatives, may not fully account for the long-range dependencies and spatial diffusion patterns observed in real-world substance abuse cases. To address these limitations, a more comprehensive framework is needed. One that integrates non-local effects to better reflect the progression of addiction, the role of external interventions, and the impact of social and environmental influences.

2 Fractional operators and global rate of change

Here, we present some well known fractional operators.

Definition 2.1. *The Riemann-Liouville fractional-order integral operator is* [22]

$$I_t^n f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{n-1} f(\tau) d\tau. \tag{2.1}$$

Definition 2.2. *The Riemann-Liouville fractional-order derivative is defined as*

$$D_t^{\alpha} f(t) = \frac{1}{\Gamma(k-\alpha)} \frac{d^k}{dx^k} \int_0^t (t-\mu)^{k-\alpha-1} f(\mu) d\mu. \tag{2.2}$$

As an alternative to the Riemann-Liouville fractional derivative, the Caputo-Fabrizio fractional derivative was introduced to overcome some of its limitations [11]. However, some scholars do not accept the use of fractional derivatives with non-singular kernels [13].

Definition 2.3. *Let $f \in H^1(a,b)$, $b > a$, $\alpha \in [0,1]$. The Caputo-Fabrizio derivative of fractional order α is defined as*

$${}_a^{CF}D_t^{\alpha} f(t) = \frac{M(\alpha)}{1-\alpha} \int_a^t f^{'}(\tau) \exp\left[-\alpha \frac{t-\tau}{1-\alpha}\right] d\tau. \tag{2.3}$$

Here, $M(\alpha)$ is such that $M(0) = M(1) = 1$

Definition 2.4. *Supposed that the function does not belong to $H^1(a,b)$. The Caputo-Fabrizio derivative of fractional order α is defined as follows:*

$${}_a^{CF}D_t^{\alpha} f(t) = \frac{\alpha M(\alpha)}{1-\alpha} \int_a^t \big(f(t) - f(x)\big) \exp\left[-\alpha \frac{t-\tau}{1-\alpha}\right] d\tau. \tag{2.4}$$

Again, $M(\alpha)$ is such that $M(0) = M(1) = 1$.

Definition 2.5. *The Caputo-Fabrizio fractional integral of order α is defined as*

$$I_{\alpha}^t\big(f(t)\big) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} f(t) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_a^t \big(f(t) - f(x)\big) d\tau, \tag{2.5}$$

$$M(\alpha) = \frac{2}{2-\alpha}, \quad 0 \le \alpha \le 1. \tag{2.6}$$

The Atangana-Baleanu fractional derivatives were introduced in 2016 by Atangana and Baleanu [8], they proposed the following definitions.

Definition 2.6. *Let $f \in H^1(a_1,a_2)$, $a_2 > a_1$, $\alpha \in [0,1]$, the Atangana-Baleanu fractional derivative in Caputo sense is*

$${}_a^{ABC}D_t^{\alpha} f(t) = \frac{B(\alpha)}{1-\alpha} \int_a^t E_{\alpha}\left[-\alpha \frac{(t-\tau)^{\alpha}}{1-\alpha}\right] f^{'}(\tau) d\tau. \tag{2.7}$$

Definition 2.7. *Let $f \in H^1(a_1,a_2), a_2 > a_1, \alpha \in [0,1]$ not necessary differentiable, the Atangana-Baleanu fractional derivative in Riemman-Liouville sense is given as*

$${}_a^{ABR}D_t^{\alpha} f(t) = \frac{B(\alpha)}{1-\alpha} \frac{d}{dt} \int_a^t E_{\alpha}\left[-\alpha \frac{(t-\tau)^{\alpha}}{1-\alpha}\right] f(\tau) d\tau. \tag{2.8}$$

2.1 Global rate of change

Nowadays the concept non-local operators is widely used in the field of mathematics [4, 16]. With that being said it calls for those operators to be timorously visited to improve and modify them for a better understanding of the changing world around us. We know that if $y = f(x)$ then dy/dx will be the rate of change of the variable y with respect to x. In other words, the rate of change,

$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}. \tag{2.9}$$

We can think of the above (2.9) as a proportion of a continuous function $f(x)$ between x_1 and x_2 and the function $g(t) = t$. we can extend the idea for any function of $g(x)$

$$m = \frac{f(x_2) - f(x_1)}{g(x_2) - g(x_1)}. \tag{2.10}$$

Definition 2.8. *Let f be a continuous function and $g(x)$ be a positive non-constant and increasing function, such that if $a < b$, then $g(a) < g(b)$. The global rate of change between a and b is given by*

$$M = \frac{f(b) - f(a)}{g(b) - g(a)}. \tag{2.11}$$

Definition 2.9. *Let f be a continuous function and $g(x)$ be a positive continuous increasing function in the interval $[a, b]$ and non-zero for all $t \in [a, b]$. The derivative of f with respect to the function g is given by*

$$D_g f(t) = \lim_{t \to t_1} \frac{f(t) - f(t_1)}{g(t) - g(t_1)}. \tag{2.12}$$

When $g(t) = t$,

$$D_g f(t_1) = \lim_{t \to t_1} \frac{f(t) - f(t_1)}{t - t_1}, \tag{2.13}$$

we then recover the classical differential operator. To recover the fractal derivative we let $g(t) = t^\alpha$

$$D_g f(t_1) = \lim_{t \to t_1} \frac{f(t) - f(t_1)}{t^\alpha - t_1^\alpha}, \quad t > 0, \quad \alpha > 0. \tag{2.14}$$

If $g(t) = t^{2-\alpha}/(2-\alpha)$, we get

$$D_g f(t_1) = \lim_{t \to t_1} \frac{f(t) - f(t_1)}{t^{2-\alpha} - t_1^{2-\alpha}}(2-\alpha), \quad t > 0, \quad 1 \leq \alpha < 2. \tag{2.15}$$

Also, if $g(t) = t^{\beta(t)}$,

$$D_g f(t_1) = \lim_{t \to t_1} \frac{f(t) - f(t_1)}{t^{t^{\beta(t)}} - t_1^{t^{\beta(t)}}}. \tag{2.16}$$

If f and g are differentiable, then for $g(t) = t$, we get

$$D_t f(t_1) = \lim_{t \to t_1} \frac{f(t) - f(t_1)}{t - t_1} = f'(t_1).$$

For $g(t) = t^\alpha$ we get

$$\begin{aligned} D_{t^\alpha} f(t_1) &= \lim_{t \to t_1} \frac{f(t) - f(t_1)}{t^\alpha - t_1^\alpha} \\ &= \lim_{t \to t_1} \frac{f(t) - f(t_1)}{t^\alpha - t_1^\alpha} \times \frac{t - t_1}{t - t_1} \\ &= \lim_{t \to t_1} \frac{f(t) - f(t_1)}{t - t_1} \times \frac{t - t_1}{t^\alpha - t_1^\alpha} \\ &= f'(t_1) \times \frac{t_1^{1-\alpha}}{\alpha}. \end{aligned}$$

For $g(t) = t^{2-\alpha}/(2-\alpha)$, we get

$$\begin{aligned} D_g f(t_1) &= \lim_{t \to t_1} \frac{f(t) - f(t_1)}{t^{2-\alpha}/(2-\alpha)} - \frac{t_1^{2-\alpha}}{2-\alpha} \\ &= \lim_{t \to t_1} \frac{f(t) - f(t_1)}{t^{2-\alpha}/(2-\alpha) - t_1^{2-\alpha}/(2-\alpha)} \times \frac{t - t_1}{t - t_1} \\ &= \lim_{t \to t_1} \frac{f(t) - f(t_1)}{t - t_1} \times \frac{t - t_1}{t^{2-\alpha}/(2-\alpha) - t_1^{2-\alpha}/(2-\alpha)} \\ &= f'(t_1) \times t_1^{\alpha-1}. \end{aligned}$$

For $g(t) = t^{B(t)}$, we get

$$\begin{aligned} D_g f(t_1) &= \lim_{t \to t_1} \frac{f(t) - f(t_1)}{t^{B(t)} - t_1^{B(t_1)}} \\ &= \lim_{t \to t_1} \frac{f(t) - f(t_1)}{t^{B(t)} - t_1^{B(t_1)}} \times \frac{t - t_1}{t - t_1} \\ &= \lim_{t \to t_1} \frac{f(t) - f(t_1)}{t - t_1} \times \frac{t - t_1}{t^{B(t)} - t_1^{B(t_1)}} \\ &= f'(t_1) \times \frac{t^{-B(t_1)}}{B'(t_1)\ln(t_1) + B(t_1)/t_1}. \end{aligned}$$

Definition 2.10. *Let f(t) be a continuous function and g(t) be a non-constant increasing positive function. Let $K(t)$ be a kernel singular or non-singular for $0 < \alpha \leq 1$. A fractional global derivative in Caputo sense is given by* [6]

$${}_0^C D_g^\alpha f(t) = D_g f(t) * K(t),$$

derivative in Caputo sense is given by

$$ {}_0^{RL}D_g^{\alpha} f(t) = D_g\big(f(t) * K(t)\big). $$

*Here, * denotes the convolution. Now, if the kernel* $K(t) = t^{-\alpha}/\Gamma(1-\alpha)$*, we then have the power-law type*

$$ {}_0^{RL}D_g^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} D_g \int_0^t f(\tau)(t-\tau)^{-\alpha} d\tau, \tag{2.17} $$

and with Caputo we get

$$ {}_0^{C}D_g^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t D_g f(\tau)(t-\tau)^{-\alpha} d\tau. \tag{2.18} $$

To recover the Caputo-Fabrizio, we take $K(t) = \exp[-\alpha t/(1-\alpha)]/(1-\alpha)$

$$ {}_0^{CF}D_g^{\alpha} f(t) = \frac{M(\alpha)}{1-\alpha} \int_0^t D_g f(\tau) \exp\left[-\frac{\alpha}{1-\alpha}(t-\tau)\right] d\tau, \tag{2.19} $$

$$ {}_0^{CF}D_g^{\alpha} f(t) = \frac{M(\alpha)}{1-\alpha} D_g \int_0^t f(\tau) \exp\left[-\frac{\alpha}{1-\alpha}(t-\tau)\right] d\tau. \tag{2.20} $$

We recover the Atangana-Baleanu derivative for $K(t) = AB(\alpha)/(1-\alpha)\, E_{\alpha}[-\alpha t^{\alpha}/(1-\alpha)]$

$$ {}_0^{ABC}D_g^{\alpha} f(t) = \frac{AB\alpha)}{1-\alpha} \int_0^t D_g f(\tau) E_{\alpha}\left[-\frac{\alpha}{1-\alpha}(t-\tau)^{\alpha}\right] d\tau, \tag{2.21} $$

$$ {}_0^{ABR}D_g^{\alpha} f(t) = \frac{AB(\alpha)}{1-\alpha} D_g \int_0^t f(\tau) E_{\alpha}\left[-\frac{\alpha}{1-\alpha}(t-\tau)^{\alpha}\right] d\tau. \tag{2.22} $$

3 Model formulation

The population is divided into six compartments: $S(t)$ is the susceptible individuals, these are individuals who are not taking drugs, but they leave among drug users. $D(t)$ are individuals who takes drugs at times but not addicted (casual drug users). $H(t)$ is the group of individuals who are taking drugs more often and are addicted and $P(t)$ as those individuals taking drugs (addicted and not addicted) and are in a correctional service center due to drug related crimes. Individuals suffering mental illness due to drug abuse are denoted by $C(t)$ and finally, $R(t)$ denotes those individuals that are recovering and going to rehabilitation centers. Thus, the total population,

$$ N(t) = S(t) + D(t) + H(t) + C(t) + R(t). \tag{3.1} $$

We assume a constant size population with recruitment and a natural death rate given by μ. The parameter β is the strength of interactions between the susceptible individuals, casual drug users and drug addicts, that is, the influence of $D(t)$ and $H(t)$ on $S(t)$; κ is a modification factor which accounts for the increased likelihood of heavy

illicit drug users $H(t)$ to influence more new drug users compared to casual drug users. The parameter α is the rate at which casual drug users become heavy users; $\gamma, \varepsilon_1, \varepsilon_2$ and ρ denote the rates of detection and rehabilitation of drug users in the group, $D(t), H(t), P(t)$ and $C(t)$; σ and θ denote the rates at which the casual and heavy illicit drug users develop mental illness; ψ_1, ψ_2, ψ_3 and ψ_4 denote the permanent exit rates of casual, addicts and those in correctional services, due to either cessation or drug use-related death.

Also, λ_1 and λ_2 denotes the rate at which the casual and heavy illicit drug users becomes identified and go to a correctional service facility. Individuals in rehabilitation recover at rate ω and are assumed to permanently exit the model. Mentally ill individuals permanently exit the model at rate δ due to drug use-related death. Further, we assume that the mentally ill population does not influence the susceptible individuals to become illicit drug users.

The model mechanism based on the above assumptions is then

$$\frac{dS}{dt} = \pi - \frac{\beta(D + \kappa H)S}{N} - \mu S + \omega R, \tag{3.2}$$

$$\frac{dD}{dt} = \frac{\beta(D + \kappa H)S}{N} - (\alpha + \gamma + \sigma + \mu + \lambda_1 + \psi_1)D, \tag{3.3}$$

$$\frac{dH}{dt} = \alpha D - (\rho + \theta + \mu + \psi_2 + \lambda_2)H, \tag{3.4}$$

$$\frac{dP}{dt} = \lambda_1 D + \lambda_2 H - (\epsilon_1 + \tau + \mu + \psi_3)P, \tag{3.5}$$

$$\frac{dC}{dt} = \sigma D + \theta H + \tau P - (\epsilon_2 + \mu + \psi_4 + \phi)C, \tag{3.6}$$

$$\frac{dR}{dt} = \gamma D + \rho H + \epsilon_2 C + \epsilon_1 P + \phi C - (\mu + \omega)R \tag{3.7}$$

with the initial conditions given as

$$\begin{aligned} &S(0) = S_0, \quad D(0) = D_0, \quad H(0) = H_0, \\ &P(0) = P_0, \quad C(0) = C_0, \quad R(0) = R_0. \end{aligned} \tag{3.8}$$

Using the assumption made above, the schematic diagram is given in Fig. 1.

$$N(t) = S(t) + D(t) + H(t) + P(t) + C(t) + R(t). \tag{3.9}$$

Differentiating (3.9) with respect to t, we get

$$\begin{aligned} \frac{dN(t)}{dt} &= \frac{dS(t)}{dt} + \frac{dH(t)}{dt} + \frac{D(t)}{dt} + \frac{P(t)}{dt} + \frac{C(t)}{dt} + \frac{R(t)}{dt} \\ &= \pi - \mu N + \psi_1 D + \psi_2 H + \psi_3 P + \psi_4 C, \end{aligned} \tag{3.10}$$

$$\frac{dN(t)}{dt} \leq \pi - \mu N. \tag{3.11}$$

This is an ordinary differential equation, which has the solution

$$N(t) \leq \frac{\pi}{\mu} + \left(N_0 - \frac{\pi}{\mu}\right) e^{-\mu t}. \tag{3.12}$$

Furthermore,

$$N(t) \leq \frac{\pi}{\mu} + \left(N_0 - \frac{\pi}{\mu}\right) e^{-\mu t} \leq \frac{\pi}{\mu}. \tag{3.13}$$

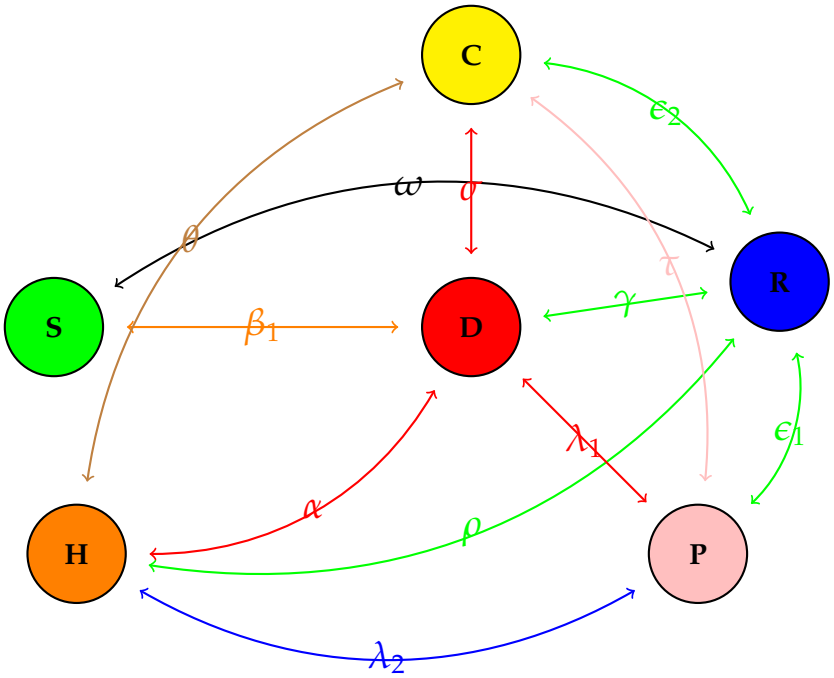


Figure 1: Flow diagram, with $\beta_1 = \beta(D + \kappa H)/N$.

4 Positiveness and boundness of solutions

Assuming that the initial conditions are positive for all $t \geqslant 0$, we show that the system remains positive. Firstly, we consider the $D(t)$ class

$$\begin{aligned} \frac{dD}{dt} &= \frac{\beta(D + kH)}{N} S - (\alpha + \gamma + \sigma + \lambda_1 + \psi_1) D \\ &\geqslant -(\alpha + \gamma + \sigma + \lambda_1 + \psi_1) D, \qquad \forall\, t \geqslant 0 \\ &\Rightarrow \quad D(t) \geqslant D_0 e^{-(\alpha+\gamma+\sigma+\lambda_1+\psi_1)t}, \quad \forall\, t \geqslant 0, \end{aligned}$$

$$\begin{aligned}\frac{dH}{dt} &= \alpha D - (\rho+\theta+\mu+\psi_2+\lambda_2)H \\ &\geqslant -(\rho+\theta+\mu+\psi_2+\lambda_2)H, \qquad \forall\, t \geqslant 0, \\ &\Rightarrow \quad H(t) \geq H_0 e^{-(\rho+\theta+\mu+\psi_2+\lambda_2)t}, \quad \forall\, t \geqslant 0.\end{aligned}$$

Using the same procedure

$$\begin{aligned}P(t) &\geqslant P_0 e^{-(\varepsilon_1+\tau+\mu+\psi_3)t}, \quad \forall\, t \geqslant 0, \\ C(t) &\geqslant C_0 e^{-(\varepsilon_2+\mu+\phi+\psi_4)t}, \quad \forall\, t \geqslant 0, \\ R(t) &\geqslant R_0 e^{-(\mu+\omega)t}, \qquad \forall\, t \geqslant 0,\end{aligned}$$

$$\begin{aligned}S(t) &= \pi - \frac{\beta(D+kH)}{N} - \mu S + \omega R, \qquad \forall\, t \geqslant 0, \\ S(t) &\geqslant -\left(\frac{\delta(D+kH)}{N} + \mu\right) S, \qquad \forall\, t \geq 0 \\ &\geqslant -\left(\frac{\beta}{N}(kH+D) + \mu\right) S, \quad (Ht \geqslant 0) \\ &\geqslant -\left(\frac{\beta}{N}(k|H|+|D|) + \mu\right) S \\ &\geq -\left(\frac{\beta}{N}\left(k \sup_{t\in[t_0,T]} |H| + \sup_{t\in[t_0,T]} |D|\right) + \mu\right) S \\ &\geqslant -\left(\frac{\beta}{N}\left(k\|H\|_\infty + \|D\|_\infty\right) + \mu\right) S, \quad \forall\, t \geq 0, \\ S(t) &\geqslant S_0 e^{-(\frac{\beta}{N}(k\|H\|_\infty+\|D\|_\infty)+\mu)t}, \qquad \forall\, t \geq 0.\end{aligned}$$

We can now define the following feasible region:

$$\Omega = \left\{(S,D,H,P,C,R) \in \mathbb{R} \,:\, 0 \leq S+D+H+P+C+R = N \leq \frac{\pi}{\mu}\right\}.$$

4.1 Reproduction number

The reproduction number, also known as the basic reproduction number or R_0, is a measure of the average number of secondary cases generated by a single infected individual in a population that is entirely susceptible to a disease, [14, 18]. In the context of a drug use mathematical model, the reproduction number would represent the average number of individuals who start using a drug as a result of one individual already using the drug.

We use the next generation matrix to derive the reproduction number using the following system:

$$\begin{aligned}
\frac{dD}{dt} &= \frac{\beta}{n}(D+kH)S - (\alpha+\gamma+\sigma+\lambda_1+\psi_1)D,\\
\frac{dH}{dt} &= \alpha D - (\rho+\theta+\mu+\psi_2+\lambda_2)H,\\
\frac{dP}{dt} &= \lambda D + \lambda_2 H - (\varepsilon_1+\tau+\mu+\psi_3)P,\\
\frac{dC}{dt} &= \sigma D + \theta H + \tau P - (\varepsilon_2+\mu+\phi+\psi_4)C.
\end{aligned}$$

Use above equation, we have

$$F = \begin{bmatrix} 0 & \frac{\beta}{N}\left(\frac{\pi}{\mu}\right) & 0 & \frac{\beta}{N}k\left(\frac{\pi}{\mu}\right) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$F = \begin{bmatrix} \alpha+\gamma+\sigma+\lambda_1+\psi_1 & 0 & 0 & 0 \\ -\alpha & \rho+\theta+\mu+\psi_2+\lambda_2 & 0 & 0 \\ -\lambda_1 & -\lambda_2 & \varepsilon_1+\tau+\mu+\psi_3 & 0 \\ -\sigma & -\phi & -\tau & \varepsilon_2+\mu+\phi+\psi_4 \end{bmatrix}.$$

Thus, the reproduction number is given by

$$R_0 = \frac{\pi}{\mu}\left(\frac{\beta}{N}\right)\left[\frac{\sigma\big(k(\rho+\theta+\mu+\lambda_2+\varphi_2)\big)}{A_1A_2A_4} + \frac{\alpha}{A_1A_2}\right],$$

where

$$\begin{aligned}
A_1 &= \alpha+\gamma+\sigma+\lambda_1+\psi_1,\\
A_2 &= \rho+\theta+\mu+\lambda_2+\psi_2,\\
A_4 &= \varepsilon_2+\mu+\phi+\psi_4.
\end{aligned}$$

5 Steady states analysis

$$\frac{dD}{dt} = \frac{dS}{dt} = \frac{dH}{dt} = \frac{dP}{dt} = \frac{dC}{dt} = \frac{dR}{dt} = 0. \tag{5.1}$$

5.1 Equilibrium points

Here we investigate all the possible equilibrium points.
Case 1: If $S = 0$,

$$D = 0, \quad H = 0, \quad P = 0, \quad C = 0, \quad \text{or} \quad R = 0.$$

Case 2: If $D, H, C, P, R = 0$,

$$E_0^* = \left(\frac{\pi}{\mu}, 0, 0, 0, 0, 0\right) \quad \text{drug-free equilibrium.}$$

Case 3: If S, D, H, P, C and R are greater than 0,

$$\pi - \mu S - \omega R - d_1 D = 0,$$
$$D = \frac{h_1 H}{\alpha},$$
$$H\left(\frac{\lambda h_1}{\alpha} + \lambda_2\right) - p_1 P = 0 \;\Rightarrow\; P = \left(\frac{\lambda h_1/\alpha + \lambda_2}{p_1}\right) H,$$
$$\frac{h_1 H}{\alpha} + \phi H + \tau \frac{(\lambda h_1/\alpha + \lambda_2)}{p_1} H - c_1 C = 0 \;\Rightarrow\; C = \left[\frac{h_1/\alpha + \phi + \tau(\lambda h_1/\alpha + \lambda_2)/p_1}{c_1} H\right],$$
$$H\left(\frac{\gamma}{\alpha} h_1 + \rho + (\varepsilon_2 + \phi)\left[\frac{h_1}{\alpha} + \phi + \tau\left(\frac{\lambda h_1/x + \lambda_2}{P_1}\right)\right] + \frac{\varepsilon_2(\lambda h_1/\alpha + \lambda_2)}{p_1}\right) - (\mu + \omega) R = 0.$$

Also

$$\left(\beta \frac{(h_1 H/\alpha + kH)}{N}\right) S - \frac{d_1 h_1}{\alpha} H = 0,$$
$$H\left(\beta(h_1/\alpha + k) S - \frac{N d_1 h_1}{\alpha}\right) = 0.$$

Thus,

$$H = 0 \quad \text{or} \quad S^* = N d_1 h_1 \left(\frac{\alpha}{\beta(h_1/\alpha + k)}\right)^{-1},$$
$$\pi - \frac{\mu(d_1 h_1 N/\alpha)}{\beta(h_1/\alpha + k)} - \left(\omega - \frac{h_1 d_1}{\alpha}\left(\frac{\mu + \omega)}{k_1}\right)\right) R = 0, \tag{5.2}$$
$$R^* = \left(\pi - \frac{\mu(\alpha h_1/\alpha)}{\beta(h_1 d_1/\alpha + k)}\right)\left(\omega - \frac{h_1 d_1}{\alpha}\left(\frac{\mu + \omega}{k_1}\right)\right)^{-1}, \tag{5.3}$$

where

$$k_1 = \frac{\gamma}{\alpha} h_1 + \rho + (\varepsilon_2 + \phi)\left[\frac{h_1}{\alpha} + \phi + \tau\left(\frac{\lambda h_1}{\alpha} + \lambda_2\right)\right] + \frac{\varepsilon_2(\lambda h_1/\alpha + \lambda_2)}{p_1}, \tag{5.4}$$
$$H^* = \frac{\mu + \omega}{k_1} R = \frac{\mu + \omega}{k_1} R^*, \tag{5.5}$$

$$D^* = \frac{h_1}{\alpha}\left(\frac{\mu+\omega}{k_1}\right)R^*, \tag{5.6}$$

$$C^* = \left[\frac{\left(h_1/\alpha+\phi+\tau\left((\lambda h_1/\alpha+\lambda_2)/p_1\right)\right.}{c_1}\right)H^*, \tag{5.7}$$

$$P^* = \frac{(\lambda h_1/\alpha+\lambda_2)}{p_1}H^*, \tag{5.8}$$

$$E_1^* = (S^*, D^*, H^*, P^*, C^*, R^*) \quad \text{endemic equilibrium.} \tag{5.9}$$

6 Stability analysis

Here we will study the stability nature of our two equilibrium points (E_0, E_*), we first discuss the drug-free equilibrium point then later the endemic point the Jacobian matrix for our model is given by

$$\mathrm{J} = \begin{pmatrix} -\mu & -\frac{\beta}{N}S & -\frac{\beta}{N}kS & 0 & 0 & \omega \\ 0 & -\left(A_1+\frac{\beta}{N}S\right) & -\frac{\beta}{N}kS & 0 & 0 & 0 \\ 0 & \alpha & -A_2 & 0 & 0 & 0 \\ 0 & \lambda_1 & \lambda_2 & -A_3 & 0 & 0 \\ 0 & \sigma & \phi & \tau & -A_4 & 0 \\ 0 & \gamma & \rho & \epsilon_1 & \theta+\epsilon_2 & -(\mu+\omega) \end{pmatrix}.$$

6.0.1 Local stability drug-free equilibrium point

The Jacobian drug-free equilibrium $J(E_0)$

$$\mathrm{J(E_0)} = \begin{pmatrix} -\mu & 0 & & 0 & 0 & \omega \\ 0 & -A_1 & & 0 & 0 & 0 \\ 0 & \alpha & -A_2 & 0 & 0 & 0 \\ 0 & \lambda_1 & \lambda_2 & -A_3 & 0 & 0 \\ 0 & \sigma & \phi & \tau & -A_4 & 0 \\ 0 & \gamma & \rho & \epsilon_1 & \theta+\epsilon_2 & -(\mu+\omega) \end{pmatrix}.$$

The trace and the determinant of the Jacobian matrix are

$$\mathrm{tr}\big(J(E_0)\big) = -(\mu + A_1 + A_2 + A_3 + A_4 + (\mu+\omega)) < 0,$$

$$\mathrm{Det}\big(J(E_0)\big) = \big(\mu \times A_1 \times A_2 \times A_3 \times A_4 \times (\mu+\omega)\big) > 0.$$

We can conclude that the drug-free equilibrium E_0 of the model is locally asymptotically stable if the $\mathrm{tr}(J(E_0)) < 0$ and $\mathrm{Det}(J(E_0)) > 0$.

Theorem 6.1. *The Drug-free equilibrium of the proposed model for illicit drug abuse is globally asymptotically Stable within the feasible region if $R_0 < 1$ and unstable if $R_0 > 1$.*

Proof. Lets consider the following Lyapunov function:

$$L = \frac{1}{\alpha+\gamma+\sigma+\lambda_1+\psi_1}D + \frac{1}{\rho+\phi+\mu+\alpha_2+\psi_2}H + \frac{1}{\varepsilon_1+\tau+\mu+\psi_3}P + \frac{1}{\varepsilon_2+\mu+\theta+\psi_4}C$$

such that

$$\begin{aligned}\frac{dL}{dt} &= \frac{1}{\alpha+\gamma+\sigma+\lambda_1+\psi_1}\frac{dD}{dt} + \frac{1}{\rho+\phi+\mu+\lambda_2+\psi_2}\frac{dH}{dt} \\ &\quad + \frac{1}{\varepsilon_1+\tau+\mu+\psi_3}\frac{dP}{dt} + \frac{1}{\varepsilon_2+\mu+\theta+\psi_4}\frac{dC}{dt} \\ &= \frac{1}{\alpha+\gamma+\sigma+\lambda_1+\psi_1}\left[\frac{\beta}{N}(D+KH)S - (\alpha+\gamma+\sigma+\lambda_1+\psi_1)D\right] \\ &\quad + \frac{1}{\rho+\theta+\mu+\psi_2}\left[\alpha D - (\rho+\theta+\mu+\psi_2)H\right] \\ &\quad + \frac{1}{\varepsilon_1+\tau+\mu+\psi_3}[\lambda_1 D + \lambda_2 H - (\varepsilon_1+\tau+\mu+\psi_3)P] \\ &\quad + \frac{1}{\varepsilon_2+\mu+\phi+\psi_4}\left[\sigma D + \phi H + \tau P - (\varepsilon_2+\mu+\phi+\psi_4)C\right].\end{aligned}$$

Then simplifying, we get

$$\begin{aligned}\frac{dL}{dt} &= \frac{\beta(D+kH)S}{N\left(\alpha+\gamma+\sigma+\lambda_1+\psi_1\right)} - D + \frac{\alpha D}{\rho+\phi+\mu+1+\psi_2} - H \\ &\quad + \frac{\lambda_1 D+\lambda_2 H}{\varepsilon_1+\tau+\mu+\varphi_3} - P + \frac{\theta D+\psi H+\tau P}{\varepsilon_2+\mu+\psi+\varphi_4} - C, \\ &= \frac{\beta(D+kH)S}{N(\alpha+\gamma+\sigma+\lambda_1+\psi_1)} + \frac{\alpha D}{\rho+\phi+\mu+1+\psi_2} \\ &\quad + \frac{\lambda_1 D+\lambda_2 H}{\varepsilon_1+\tau+\mu+\varphi_3} + \frac{\theta D+\psi H+\tau P}{\varepsilon_2+\mu+\psi+\varphi_4} - [D+H+P+C].\end{aligned}$$

Also

$$\begin{aligned}\frac{dL}{dt} &= \left[\frac{\beta(D+kH)S}{N(\alpha+\gamma+\sigma+\lambda_1+\psi_1)} + \frac{\alpha D}{\rho+\phi+\mu+1+\psi_2}\right. \\ &\quad \left. + \frac{\lambda_1 D+\lambda_2 H}{\varepsilon_1+\tau+\mu+\varphi_3} + \frac{\theta D+\psi H+\tau P}{\varepsilon_2+\mu+\psi+\varphi_4}\right]\frac{1}{D+H+P+C} - [D+H+P+C] \\ &\leqslant \left\{\frac{\pi}{\mu}\frac{\beta}{N}\left[\frac{(1-\rho)\alpha(\phi+K(\varepsilon_1+\tau\tau_1+\psi_3)}{(\alpha_1\gamma+\sigma+t_1+\varphi_1)(p+\phi+\mu+\gamma_2+\varphi_2)(\varepsilon_1+\tau+\mu+\varphi_3)}\right.\right. \\ &\quad \left.\left. + \frac{\rho_\alpha}{(\alpha+\gamma+\sigma+H_1+\varphi_1)(\varepsilon_1+\tau+\mu_+\varphi_3)}\right]\frac{1}{D+H+C+P} - 1\right\}(D+H+P+C) \\ &\leq (D+H+C+P) \leq 0,\end{aligned}$$

if $R_0 \leqslant 1$,

$$D + H + C + P > 0, \quad \forall\, t.$$

As we assumed initially that all parameters are positive, then dL/dt decreases if $R_0 < 1$ and increases if $R_0 > 0$. However, when $L = 0$, then $D = H = P = C = 0$.

Let T be the biggest compact invariant set in $\{D, H, P, C \in \Omega : dL/dt \leqslant 0\}$ is E_0. Using Lasalle's invariant principle, all the solution of the illicit drug use model with data in Ω yield E_0 when $t \to \infty$ and if $R_0 \leq 1$ Therefore E_0 is globally asymptotically stable. □

6.1 Local stability endemic equilibrium point

The Jacobean matrix for the endemic equilibrium point is

$$\mathrm{J(E_*)} = \begin{pmatrix} \lambda+\mu & -\frac{\beta}{N}S & -\frac{\beta}{N}kS & 0 & 0 & \omega \\ 0 & \lambda + \left(A_1 + \frac{\beta}{N}S\right) & -\frac{\beta}{N}kS & 0 & 0 & 0 \\ 0 & \alpha & \lambda + A_2 & 0 & 0 & 0 \\ 0 & \lambda_1 & \lambda_2 & \lambda + A_3 & 0 & 0 \\ 0 & \sigma & \phi & \tau & \lambda + A_4 & 0 \\ 0 & \gamma & \rho & \epsilon_1 & \theta + \epsilon_2 & \lambda + (\mu + \omega) \end{pmatrix}.$$

We have the following characteristic polynomial:

$$\begin{aligned} P(\lambda) = {} & (\mu+\lambda)(A_3+\lambda)(A_4+\lambda)\big((\mu+\omega)+\lambda\big) \\ & \times \left[A_2\left(A_1 + \frac{\beta}{N}S_*\right) + \lambda\left(A_1 + \frac{\beta}{N}S_*\right) + \frac{\alpha\beta}{N}kS_* + \lambda A_2 + \lambda^2\right], \end{aligned}$$

which we can simplify to this form

$$P(\lambda) = \lambda^6 + l_1\lambda^5 + l_2\lambda^4 + l_3\lambda^3 + l_4\lambda^4 + l_5\lambda^2 + l_6\lambda$$

using the following equality:

$$P = \det|I_M\lambda - JE_*| = 0,$$

I_M is a 4×4 unit matrix.

The polynomials above has the following square Routh-Hurwitz matrix:

$$Z = \begin{bmatrix} l_1 & l_3 & l_5 & 0 & 0 & 0 \\ 1 & l_2 & l_4 & l_6 & 0 & 0 \\ 0 & l_1 & l_3 & l_5 & 0 & 0 \\ 0 & 1 & l_2 & l_4 & l_6 & 0 \\ 0 & 0 & l_1 & l_3 & l_5 & 0 \\ 0 & 0 & 1 & l_2 & l_4 & l_6 \end{bmatrix}.$$

The condition for stability is as follows:

$$
\begin{aligned}
Z_1 &= l_1 > 0,\\
Z_2 &= l_1 l_2 - l_3 > 0\\
Z_3 &= -l_1^2 l_4 + l_1 l_2 l_3 + l_1 l_5 - l_3^2 > 0,\\
Z_4 &= l_1^2 l_2 l_6 - l_1^2 l_4^2 - l_1 l_2^2 l_5 + l_1 l_2 l_3 l_4 - l_1 l_3 l_6 + 2 l_1 l_4 l_5 + l_2 l_3 l_5 - l_3^2 l_4 - l_5^2 > 0,\\
Z_5 &= -l_1^3 l_6^2 + 2 l_1^2 l_2 l_5 l_6 + l_1^2 l_3 l_4 l_6 - l_1 l_2^2 l_5^2 - l_1 l_2 l_3^2 l_6 + l_1 l_2 l_3 l_4 l_5 - 3 l_1 l_3 l_5 l_6\\
&\quad + 2 l_1 l_4 l_5^2 + l_2 l_3 l_5^2 - l_1^2 l_4^2 l_5 + l_3^3 l_6 - l_3^2 l_4 l_5 - l_5^3 > 0,\\
Z_6 &= l_6 Z_5 > 0.
\end{aligned}
$$

Theorem 6.2. *If $R_0 \geq 1$, the drug abuse endemic equilibrium point is globally asymptotically stable.*

Proof. We make use of the Lyapunov function

$$
\begin{aligned}
&L\left(S^*, D^*, H^*, P^*, C^*, R^*\right)\\
&= \left(S - S^* - S^* \log \frac{S*}{S}\right) + \left(D - D^* - D^* \log \frac{D^*}{D}\right)\\
&\quad + \left(H - H^* - H^* \log \frac{H^*}{H}\right) + \left(P + P^* - P^* \log \frac{P^*}{P}\right)\\
&\quad + \left(C - C^* - C^* \log \frac{C^*}{C}\right) + \left(R - R^* - R^* \log \frac{R^*}{R}\right).
\end{aligned}
$$

Derivating L with respect to t, gives

$$
\begin{aligned}
\frac{dL}{dt} = L &= \left(\frac{S - S^*}{S}\right)\frac{dS}{dt} + \left(\frac{D - D^*}{D}\right)\frac{dD}{dt} + \left(\frac{H + H^*}{H}\right)\frac{dH}{dt}\\
&\quad + \left(\frac{P - P^*}{P}\right)\frac{dP}{dt} + \left(\frac{C - C^*}{C}\right)\frac{dC}{dt} + (R - R^*)\frac{dR}{dt}.
\end{aligned}
$$

Substituting $dS/dt, dD/dt, dH/dt, dC/dt, dR/dt$,

$$
\begin{aligned}
\frac{dL}{dt} &= \left(\frac{S - S^x}{S}\right)\left[\pi - \frac{\beta}{N}(D + 1 + k)S - \mu S + wR\right]\\
&\quad + \left(\frac{D + D^*}{D}\right)\left[\frac{\beta}{N}(D + kH) - (\alpha + \gamma + \sigma + \lambda_1 + \psi_1)D\right]\\
&\quad + \left(\frac{H + H^*}{H}\right)[\alpha D - (\rho + \theta + \mu + \psi_2 + \lambda_2)H]\\
&\quad + \left(\frac{P + P^*}{P}\right)[\lambda_1 D + \lambda_2 H - (\varepsilon_1 + \tau + \mu + \psi_3)P]\\
&\quad + \left(\frac{C + c^*}{c}\right)[\sigma D + H + \tau P - (\varepsilon_2 + \mu + \psi_4 + \phi)C']\\
&\quad + \left(\frac{R + R^*}{R}\right)[\gamma D + \rho H + (\phi + \varepsilon_2)C + \varepsilon_1 P - (\mu + \omega)R].
\end{aligned}
$$

Simplifying further give

$$\begin{aligned}
\frac{dL}{dt} &= \pi - \mu\frac{\beta S^*}{S} - k*H\frac{(S-S^*)^2}{S} + kH^*\frac{(S-S^*)^2}{S} - \beta D\frac{(S-S^*)^2}{S} \\
&\quad + kD^*\frac{(S-S^*)^2}{S} - \frac{(S-S^*)^2}{S}\mu - \frac{(P-P^*)^2}{P}(\epsilon_1+\psi_1+\gamma) + kS\alpha H - k*SH^* \\
&\quad + \frac{P^*}{P}k\lambda_2 S^*H - \frac{P^*}{P}k\lambda S^*H^* - \frac{P^*}{P}\beta SD + \frac{P^*}{P}\omega S^* + \frac{P^*}{P}\tau S^*D, \\
\frac{dL}{dt} &= F - \Omega.
\end{aligned}$$

$$\begin{aligned}
\Omega &= \pi + k\beta H^*\frac{(S-S^*)^2}{S} + \lambda D^*\frac{(S-S^*)^2}{S} \\
&\quad + \alpha\psi SH + \omega\psi_2 S^*H^* + \gamma SH + \tau S^*C^* + \frac{P^*}{P}\sigma\beta SD \\
&\quad + \frac{P^*}{P}\omega\mu S^*D + \frac{P^*}{P}\omega SI^* + \frac{P^*}{P}\omega S^*H + \rho\gamma C\frac{H^*}{H} \\
&\quad + \tau P + \tau\theta P^*\frac{D^*}{D} + \psi_2 D + \theta P + \varepsilon_1 D^*\frac{C^*}{C} + \tau P^*\frac{C^*}{C} \\
&\quad + \theta P^*\frac{C^*}{C} + \tau H + \lambda P^*\frac{D^*}{D} + \gamma H + \delta C \\
&\quad + \beta D + \gamma H^*\frac{R^*}{R} + (\mu\omega)C^*\frac{R^*}{R} + \omega D^*\frac{R^*}{R} + \gamma P^*, \\
F &= \lambda\frac{S^*}{S} + \omega\psi_3 D\frac{(S-S^*)^2}{S} + k\beta D\frac{(S-S^*)^2}{S} + \frac{(S-S^*)^2}{S}\mu \\
&\quad + \frac{(P-P^*)^2}{P}(\alpha+\theta+\mu) + \omega\mu SH^* + \tau KS^*C + \rho\mu S \\
&\quad + \omega SD^* + \omega S^*H + \frac{P^*}{P}\omega\mu SP + \frac{D^*}{D}\omega KS^*D^* + \frac{C^*}{C}\psi_1 SD \\
&\quad + \frac{P^*}{P}\omega S^*H^* + \frac{(H-H^*)^2}{H}(\varepsilon+\rho+\mu) + \lambda_2 P^* \\
&\quad + \beta kC\frac{D^*}{D} + \frac{(C-C)^2}{C}(\omega+\mu) + CC^* + \theta C^* \\
&\quad + \varepsilon D\frac{C^*}{C} + \theta P\frac{H^*}{H} + \frac{(D-H^*)^2}{D}(\rho+\gamma+\mu) \\
&\quad + \tau P\frac{C^*}{C} + \frac{(R-R^*)^2}{R}\mu + \gamma P^* + \tau\psi_3 P^*\frac{D^*}{D} \\
&\quad + \epsilon_1 C^* + \alpha D^* + \gamma H\frac{R^*}{R} + \beta C\frac{R^*}{R} + \gamma\frac{R^*}{R}.
\end{aligned}$$

So have that $dL/dt < 0$ if $F+ < \Omega$, however

$$0 = \Omega - \Omega \;\Rightarrow\; \frac{dL}{dt} = 0.$$

So, the largest compact invariant set for illicit drug model in the region

$$\left\{(S^*, D^*, H^*, P^*, C^*, R^*) \in \Gamma : \frac{dL}{dt} = 0\right\}.$$

It is the endemic point E_*.

We therefore conclude that E_* is globally asymptotically stable in Γ if $F < \Omega$, by Lasalle's principle. □

7 Existence and uniqueness for the illicit drug model

In this section, we proof the existence and uniqueness for the illicit drug model.

We simplify our model as follows:

$$\begin{aligned}
\frac{dS(t)}{dt} &= f_1(t, S),\\
\frac{dD}{dt} &= f_2(t, D),\\
\frac{dH}{dt} &= f_3(t, H),\\
\frac{dP}{dt} &= f_4(t, P),\\
\frac{dC}{dt} &= f_5(t, C),\\
\frac{dR}{dt} &= f_6(t, R).
\end{aligned}$$

We show that f_1, f_2, f_3, f_4, f_5 and f_6 are bounded. That is

$$\begin{aligned}
|f_1(t, S_1) - f_1(t, S_2)|^2 &\leq k_1|S_1 - S_2|^2, \quad \forall\, t \in [0, T],\\
|f_2(t, D_1) - f_2(t, D_2)|^2 &\leq k_2|D_1 - D_2|^2, \quad \forall\, t \in [0, T],\\
|f_3(t, H_1) - f_3(t, H_2)|^2 &\leq k_3|H_1 - H_2|^2, \quad \forall\, t \in [0, T],\\
|f_4(t, P_1) - f_4(t, P_2)|^2 &\leq k_4|P_1 - P_2|^2, \quad \forall\, t \in [0, T],\\
|f_5(t, C_1) - f_5(t, C_2)|^2 &\leq k_s|C_1 - C_2|^2, \quad \forall\, t \in [0, T],\\
|f_6(t, R_1) - f_6(t, R_2)|^2 &\leq k_1|R_1 - R_2|^2, \quad \forall\, t \in [0, T].
\end{aligned}$$

To show that $f_1, f_2, f_3, f_4, f_5, f_6$ are indeed Lipschitz

$$\begin{aligned}
|f_1(t, S_1) - f_1(t, S_2)|^2 &\leq \left|\frac{\beta}{N}(D + \kappa H)S_1 - \mu S_1 - \frac{\beta}{N}(D + \kappa H)S_2 + \mu_{S_2}\right|^2\\
&\quad | \leq \big(\beta(1 + k) + \mu\big)S_1 \cdot -\big(\beta(1 + k) + \mu\big)S_2\big|^2\\
&\leq |\beta(1 + k + \mu)|^2|S_1 - S_2|^2,
\end{aligned}$$

$$
\begin{aligned}
|f_2(t,D_1)-f_2(t,D_2)|^2 &\le |(1+kH)S+d_1(D_1-D_2)-\beta(1+kH)|^2\\
&\le d_1|D_1-D_2|^2 \le \bar{K}_2|D_1-D_2|^2,\\
|f_3(t,H_1)\delta-f_3(t,H_2)|^2 &= |-H_1|^2|H_1-H_2|^2\\
&\le H_1^2|H_1-H_2|^2 \le \bar{k}_3|H_1-H_2|^2,\\
|f_4(t,P_1)-f_4(t,P_2)|^2 &= |-(\varepsilon_1+\tau+\mu+\psi_1)|^2|P_1-P_2|^2\\
&\le (\varepsilon_1+\tau_+ +\mu+\psi_1)^2|P_1-P_2|^2\\
&\le K_4|P_1-P_2|^2,\\
|f_5(t,C_1)-f_5(t,C_2)|^2 &= |-(\varepsilon_2+4+\phi+\psi_4)|^2|C_1-C_2|^2\\
&\le (\varepsilon_2+\mu+\phi+\psi_4)^2|C_1-C_2|^2\\
&\le \bar{K}_s|C_1-C_2|^2,\\
|f_6(t,R_1)-f_0(t,R_2)|^2 &= |-(\mu+\omega)|^2|R_1-R_2|^2\\
&\le (\mu+\omega)^2|R_1-R_2|^2\\
&\le \bar{K}_6|R_1-R_2|^2.
\end{aligned}
$$

Then we show that the following hold:

$$
\begin{aligned}
&|f_1(t,S)|^2 \le k_1(1+|S|^2) \quad \forall\, t\in[0,T],\\
&|f_2(t,D)|^2 \le K_2(1+|D|^2) \quad \forall\, t\in[0,T],\\
&|f_3(t,H)|^2 \le k_3(1+|H|^2) \quad \forall\, t\in[0,T],\\
&|f_4(t,P)|^2 \le k_4(1+|P|^2) \quad \forall\, t\in[0,T],\\
&|f_5(t,C)|^2 \le k_5(1+|C|^2) \quad \forall\, t\in[0,T],\\
&|f_6(t,R)|^2 \le k_6(1+|R|^2) \quad \forall\, t\in[0,T].
\end{aligned}
\tag{7.1}
$$

For $f_1(t,s)$, we have

$$
\begin{aligned}
|f_1(t,S)|^2 &= \left|\pi-\frac{S}{N}(D+kH)-\mu s+wR\right|^2\\
&\le |\pi-f(D+k)s-\mu s+wR|^2\\
&\le 3\big(\pi^2+\omega^2(1+k)^2|s|^2-\mu^2|S|^2+\omega^2|R|^2\big)\\
&\le 3\big(\pi^2+\omega^2|R|^2+\big(2R^2(1+k)^2-r^2\big)|s|^2\big)\\
&\le 3\left(\pi^2+w^2\sup_{t\le T}|R|^2+\big(2\beta^2(1+k)^2-\mu^2\big)|s|^2\right)\\
&\le 3\big(\pi^2+\omega^2\|R\|_\infty+\big(2\beta^2(1+k)^2-\mu^2\big)|s|^2\big)\\
&\le \pi^2+w^2|R|_\infty\left(1+\frac{2\beta^2u+k^2-\mu^2}{\pi^2+w^2\|R\|_\infty}|s|^2\right).
\end{aligned}
$$

We require that

$$\frac{2\beta^2(1+k)^2-\mu}{\pi^2+w^2\|R\|_\infty}<1.$$

For $f_2(1,D)$

$$\begin{aligned}|f_2(t,D)|^2&=\left|\frac{\beta(0+kH)}{N}S-(\alpha+\gamma+\sigma+\phi_1+\varphi_1)D\right|^2\\&\leq 2\beta^2(1+k)^2|s|^2+d_1^2|D|^2\\&\leq 2\big(\beta^2(1+k)^2\sup_{t\leq T}|S|^2+d_1^2|D|^2\big)\\&\leq 2\big(\beta^2(1+k)^2\|S\|_\infty^2+d_1^2|D|^2\big)\\&\leq 2S^2(1+h)^2\|S\|_\infty^2\left(1+\frac{d_1^2}{2\beta^2(1+k)^3\|S\|_\infty}|D|^2\right)\\&\leq k_2(1+|D|^2),\end{aligned}$$

whenever

$$\frac{d_1^2}{\beta^2(k)^2\|S\|_\infty}<1.$$

For $f_3(t,H)$

$$\begin{aligned}|f_3(t,H)|^2&=|\alpha D-(P+\phi+\mu+\lambda_2+\psi_2)H|^2\\&\leq 2\Big(\alpha^2\sup_{t\leq T}|D|^2+(H_1)^2|H|^2\Big)\\&\leq 2\big(\alpha^2\|D\|_\alpha+h_1^2|H|^2\big)\\&\leq 2\alpha^2\|D\|_\infty\left(1+\frac{h_1H^2}{2\alpha^2\|D\|_\infty}\|H\|^2\right)\\&\leq k_3(1+|H|^2)\end{aligned}$$

when $h_1/(\alpha^2\|D\|_\infty)<1$.
For $f_4(t,P)$

$$\begin{aligned}|f_4(t,P)|^2&=|\lambda_1D+\lambda_2H-(\varepsilon_1+\tau+\mu+\varphi_3)P|^2\\&\leq 2\left(\lambda_1^2\sup_{t\in[0,T]}H^2+\lambda_2^2\sup_{t\in[0,T]}|H|\right)^2+(\varepsilon_1+\tau+\mu+\psi_3)^2\Big)\\&\leq 2\big(\lambda_1^2\|D\|_\infty+\lambda_2^2\|H\|^2+P_1P\big)\\&\leq 2\Big(\lambda_1^2\|D\|_\alpha+\lambda_2^2\|H\|^2(1+P_1P).\end{aligned}$$

If

$$\frac{\varepsilon_1+\tau+\mu+\psi_3}{2d_1^2\|D\|_\infty+\varepsilon_2^2-\ell^3\|H\|_\infty^2}<1,$$

then

$$|f_4(t,p)|^2 \leq k_4(1+|p|^2) \leq k_s(1+|C|^2).$$

Such that

$$\frac{C_1^2}{\left(\sigma^2\|D|_\infty^2 + D^2\|H\|_\infty^2 + \tau^2\|P\|_\infty^2\right.} < 1,$$

$$\begin{aligned}
|f_6(t,R)|^2 &= |\gamma D + pH + \varepsilon_2 C + \varepsilon P + \phi c - (\mu+\omega)R|^2 \\
&\leq \left(\gamma^2|D|^2 + \rho|H|^2 + (\varepsilon_1+\phi)^2|C|^2 + \varepsilon|P| + (\mu+\alpha)^2\mathbb{R}\right)^2\Big) \\
&\leq \Big(\gamma^2 \sup_{t\leq T}|D|^2 + \rho^2 \sup_{t\leq T}|H|^2 + (\varepsilon_2+\phi)^2 \sup_{t\leq T} T|C|^2 \\
&\qquad + \varepsilon^2 \sup_{t\leq T}|P|^2 + (\mu+w)^2|R|^2\Big) \\
&\leq \gamma^2\|D\|_\infty^2 + \rho^2\|H\|_\infty^2 + (\varepsilon_2+\phi)^2\|C\|_\infty^2 + \varepsilon_1^2\|P\|_\infty^2 + (\mu+\omega)^2\|R\|_\infty^2 \\
&\leq \left(\gamma^2\|D\|_\infty^2 + \rho^2\|H\|_\infty^2 + (\varepsilon_2+\phi)\right)^2\|C\|_\infty^2 + \varepsilon^2\|P\|_\infty^2\Big)
\end{aligned}$$

for

$$\frac{(\mu+\omega)^2}{\gamma^2\|D\|_\infty^2 + \rho^2\|H\|_\infty^2 + (\varepsilon_2+\phi)^2\|C\|_\infty^2 + \xi_1^2\|P\|_\infty^2} < 1, \tag{7.2}$$

Therefore, if the condition on linear growth holds such that

$$\max\{c_1, c_2, c_3, c_4, c_5, c_6\} < 1, \tag{7.3}$$

where

$$\begin{aligned}
c_1 &= \frac{2\beta^2(1+k)^2 - \mu}{\pi^2 + w^2\|R\|_\infty}, \\
c_2 &= \frac{{d_1}^2}{\beta^2(k)^2\|S\|_\infty}, \\
c_3 &= \frac{h_1}{\alpha^2\|D\|_\infty}, \\
c_4 &= \frac{\varepsilon_1 + \tau + \mu + \psi_3}{2d_1^2\|D\|_\infty + \varepsilon_2^2 - \ell^3\|H\|_\infty^2} \quad \left(2d_1^2\|D\|_\infty + \varepsilon_2^2 - \ell^3\|H\|_\infty^2 > 0\right), \\
c_5 &= \frac{1}{2(\sigma^2\|D|_\infty^2 + D^2\|H\|_\infty^2 + \tau^2\|P\|_\infty^2}, \\
c_6 &= \frac{(\mu+\omega)^2}{\gamma^2\|D\|_\infty^2 + \rho^2\|H\|_\infty^2 + (\varepsilon_2+\phi)^2\|C\|_\infty^2 + \xi_1^2\|P\|_\infty^2},
\end{aligned}$$

then the system has a unique solution.

8 Numerical approximation for illicit drug model

We now apply the numerical approximation to the model above.

8.1 Numerical solution of the model with two step Lagrange global scheme

Consider the illicit model below

$$\begin{aligned}
\frac{dS}{dt} &= F_1(t,S,D,H,P,C,R) = \pi - \frac{\beta(D+\kappa H)S}{N} - \mu S + \omega R,\\
\frac{dD}{dt} &= F_2(t,S,D,H,P,C,R) = \frac{\beta(D+\kappa H)S}{N} - (\alpha+\gamma+\sigma+\mu+\lambda_1+\psi_1)D,\\
\frac{dH}{dt} &= F_3(t,S,D,H,P,C,R) = \alpha D - (\rho+\theta+\mu+\psi_2+\lambda_2)H,\\
\frac{dP}{dt} &= F_4(t,S,D,H,P,C,R) = \lambda_1 D + \lambda_2 H - (\epsilon_1+\tau+\mu+\psi_3)P,\\
\frac{dC}{dt} &= F_5(t,S,D,H,P,C,R) = \sigma D + \theta H + \tau P - (\epsilon_2+\mu+\psi_4+\phi)C,\\
\frac{dR}{dt} &= F_6(t,S,D,H,P,C,R) = \gamma D + \rho H + \epsilon_2 C + \epsilon_1 P + \phi C - (\mu+\omega)R.
\end{aligned} \tag{8.1}$$

Given that, $S_0 = s(0), I(0) = I_0, R(0) = R_0$. Applying the above scheme

$$X_{n+1} = X_n + \frac{g(t_{n+1}) - g(t_n)}{\Delta t}\frac{h}{2}[3F(t_n, X_n) - F(t_{n-1}, X_{n-1})], \quad n = 1,2,3,\ldots, \tag{8.2}$$

where

$$\begin{aligned}
X_{n+1} &= [S_n, D_n, H_n, C_n, P_n, R_n],\\
F(t_n, X_n) &= [F_1, F_2, F_3, F_4, F_5, F_6],
\end{aligned}$$

X_1 can be calculated using the midpoint approximation that is of order 2.

8.2 Numerical solution of the model with Caputo-Fabrizio fractional derivative

Let us consider the fractional simple drug use model defined by Caputo-Fabrizio fractional derivative given by an ordinary differential equation system

$$\begin{aligned}
{}^{CF}_{a}D^{\alpha}_{g}S(t) &= F_1(t,S,D,H,P,C,R) = \pi - \frac{\beta(D+\kappa H)S}{N} - \mu S + \omega R,\\
{}^{CF}_{a}D^{\alpha}_{g}D(t) &= F_2(t,S,D,H,P,C,R) = \frac{\beta(D+\kappa H)S}{N} - (\alpha+\gamma+\sigma+\mu+\lambda_1+\psi_1)D,\\
{}^{CF}_{a}D^{\alpha}_{g}H(t) &= F_3(t,S,D,H,P,C,R) = \alpha D - (\rho+\theta+\mu+\psi_2+\lambda_2)H,\\
{}^{CF}_{a}D^{\alpha}_{g}P(t) &= F_4(t,S,D,H,P,C,R) = \lambda_1 D + \lambda_2 H - (\epsilon_1+\tau+\mu+\psi_3)P,\\
{}^{CF}_{a}D^{\alpha}_{g}C(t) &= F_5(t,S,D,H,P,C,R) = \sigma D + \theta H + \tau P - (\epsilon_2+\mu+\psi_4+\phi)C,\\
{}^{CF}_{a}D^{\alpha}_{g}R(t) &= F_6(t,S,D,H,P,C,R) = \gamma D + \rho H + \epsilon_2 C + \epsilon_1 P + \phi C - (\mu+\omega)R.
\end{aligned} \tag{8.3}$$

Thus,

$$
\begin{aligned}
X_{n+1} = X_n &+ \frac{1-\alpha}{M(\alpha)}[g'(t_n)F(t_n, X_n) - g'(t_n)F(t_{n-1}, X_{n-1})] \\
&+ \frac{\alpha}{M(\alpha)}\left[(1-\alpha) + \frac{3\alpha h}{(1-\alpha)}\right] g'(t_n)F(t_n, X_n) \\
&+ \left[(1-\alpha) + \frac{\alpha h}{(1-\alpha)} g'(t_n)F(t_{n-1}, X_{n-1})\right], \quad n = 1, 2, 3, 4, 5, 6,
\end{aligned} \tag{8.4}
$$

where

$$
\begin{aligned}
X_{n+1} &= [S_n, D_n, H_n, C_n, P_n, R_n], \\
F(t_n, X_n) &= [F_1, F_2, F_3, F_4, F_5, F_6].
\end{aligned}
$$

8.3 Numerical solution of the model with midpoint Riemann-Liouville global scheme

Let us consider the fractional illicit drug model defined by Riemann-Liouville fractional derivative given by an ordinary differential equation system with the midpoint approach

$$
\begin{aligned}
{}^{RL}_{a}D^{\alpha}_{g}S(t) &= F_1(t, S, D, H, P, C, R) = \pi - \frac{\beta(D + \kappa H)S}{N} - \mu S + \omega R, \\
{}^{RL}_{a}D^{\alpha}_{g}D(t) &= F_2(t, S, D, H, P, C, R), = \frac{\beta(D + \kappa H)S}{N} - (\alpha + \gamma + \sigma + \mu + \lambda_1 + \psi_1)D, \\
{}^{RL}_{a}D^{\alpha}_{g}H(t) &= F_3(t, S, D, H, P, C, R) = \alpha D - (\rho + \theta + \mu + \psi_2 + \lambda_2)H, \\
{}^{RL}_{a}D^{\alpha}_{g}P(t) &= F_4(t, S, D, H, P, C, R) = \lambda_1 D + \lambda_2 H - (\epsilon_1 + \tau + \mu + \psi_3)P, \\
{}^{RL}_{a}D^{\alpha}_{g}C(t) &= F_5(t, S, D, H, P, C, R) = \sigma D + \theta H + \tau P - (\epsilon_2 + \mu + \psi_4 + \phi)C, \\
{}^{RL}_{a}D^{\alpha}_{g}R(t) &= F_6(t, S, D, H, P, C, R) = \gamma D + \rho H + \epsilon_2 C + \epsilon_1 P + \phi C - (\mu + \omega)R.
\end{aligned} \tag{8.5}
$$

$$
\begin{aligned}
y\left(t_{n+1}\right) &= \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} g'(\tau) f\big(\tau, y(\tau)\big)(t_{n+1} - \tau)^{\alpha-1} d\tau \\
&= \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} \frac{g\left(t_{j+1}\right) - g\left(t_j\right)}{\Delta} f\left(\frac{t_j + t_{j+1}}{2}, \frac{y_j + y_{j+1}}{2}\right) \times \left(t_{n+1} - \tau\right) d\tau \\
&= \frac{\Delta t^{\alpha-1}}{\Gamma(\alpha+1)} \sum_{j=0}^{n} \big(g(t_{i+1}) - g(t_j)\big) \\
&\quad \times f\left(t_i + \frac{h}{2}, \frac{y_j + y_{j+1}}{2}\right) \{(n - j + 1)^{\alpha} - (n - j)^{\alpha}\}, \\
y_{n+1} &= \frac{\Delta t^{\alpha-1}}{\Gamma(\alpha+1)} \sum_{j=j}^{n-1} \big(g(t_{i+1}) - g(t_j)\big) f\left(t_i + \frac{h}{2}, \frac{y_i + y_{i+1}}{2}\right) \delta^{\alpha}_{n,j}
\end{aligned}
$$

$$+\frac{\Delta t^{\alpha-1}}{\Gamma(\alpha+1)}\big(g(t_{n+1})-g(t_n)\big)f\left(t_n+\frac{h}{2},\frac{y_n+y_{n+1}^p}{2}\right),$$

where

$$\delta_{n,j}^{\alpha}=\left\{(n-j+1)^{\alpha}-(n-j)^{\alpha}\right\},$$
$$y_{n+1}^p=\frac{\Delta t^{\alpha-1}}{\Gamma(\alpha+1)}\sum_{j=0}^{n}f(t_i,y_j)\big(g(t_{j+1})-g(t_j)\big)\delta_{n,j}^{\alpha}.$$

Then

$$X(t_{n+1})=\frac{\Delta t^{\alpha-1}}{\Gamma(\alpha+1)}\sum_{j=0}^{n-1}\left[g'(t_j)F\left(t_j+\frac{h}{2},X_j\right)\right]\left\{(n-j+1)^{\alpha}-(n-j)^{\alpha}\right\}$$
$$+g'(t_n)F\left(t_n+\frac{h}{2},X_n\right)\left\{(n-j+1)^{\alpha}-(n-j)^{\alpha}\right\}$$

with

$$X_{n+1}=[S_n,D_n,H_n,C_n,P_n,R_n],$$
$$F(t_n,X_n)=[F_1,F_2,F_3,F_4,F_5,F_6].$$

9 Numerical results

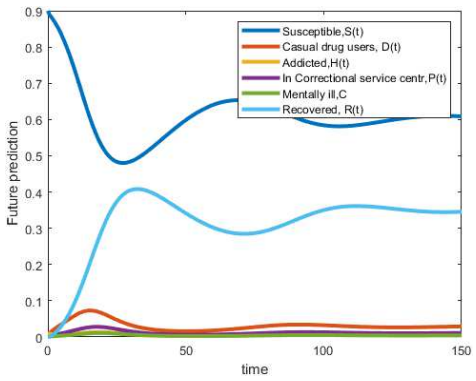
Numerical simulation involves using mathematical models and computer algorithms to simulate and analyze the behavior of physical or biological systems studied. The system is broken down into smaller parts and equations are used to describe the behavior of each part. The objective is to obtain an accurate representation of the system's behavior through time, allowing for predictions or deeper understanding of the underlying processes. We now simulate our drug abuse model with two step Lagrange scheme, two-step Lagrange Caputo-Fabrizio numerical scheme and Midpoint Riemann-Liouville global scheme. we use different step sizes, and function the $g(x)$. The following parameters were used (Table 1).

10 Conclusion

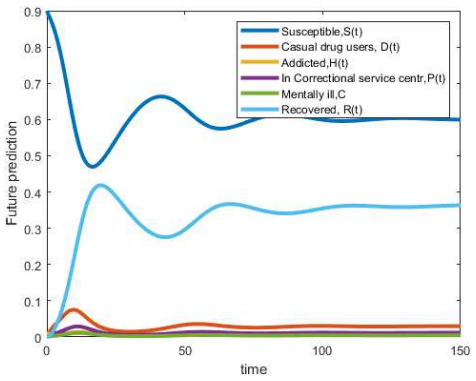
In conclusion, this study reinforces that illegal drug use continues to be a significant public health challenge on a global scale, despite ongoing research and educational initiatives. The mathematical model developed herein, which leverages global derivatives to capture non-local effects, provides a robust framework for understanding the multifaceted dynamics of illicit drug use, see Figs. 2(a), 2(b), 3(a), 3(b). By integrating both biological and social factors – including the detection of drug users-and incorporating population-specific parameters, the model is designed to reflect the diversity of drug use dynamics rather than being limited to a single substance.

Table 1: Parameter values.

Parameter	Value	Source
π	0.36	8
β	0.35	[19]
τ	0.001	8
μ	0.02	8
γ	0.01	8
ψ_1, ψ_2, ψ_3 and ψ_4	0.035	Assumed
ω	0.3	8
ρ	0.35	8
Θ	0.09	8
κ	1.25	8
α	0.01	8
ϵ_1, ϵ_2	0.6	8
σ	0.6	8

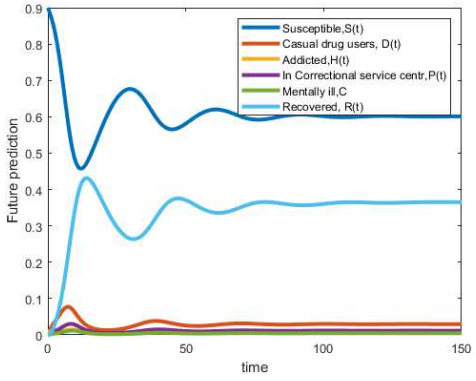


(a) $g = 0.0001x$, $h = 0.001$ for 150 days

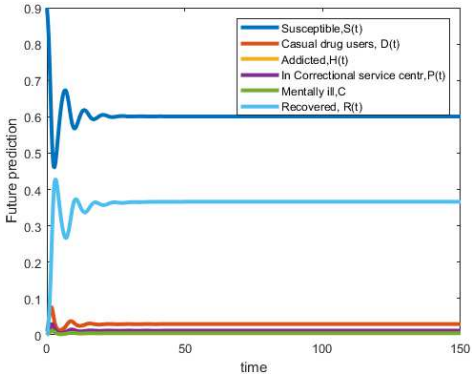


(b) $g = 0.0001x$, $h = 0.05$ for 150 days

Figure 2: Results found using the two-step Lagrange scheme.



(a) $g = x$, $h = 0.001$ for 150 days

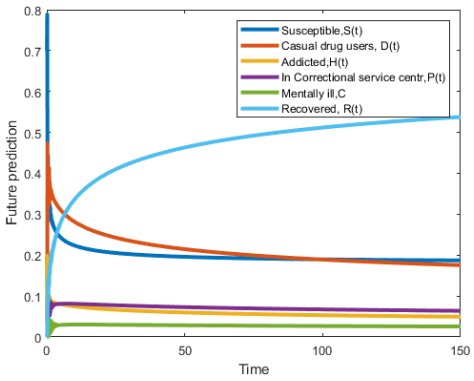


(b) $g = x$, $h = 0.05$ for 150 days

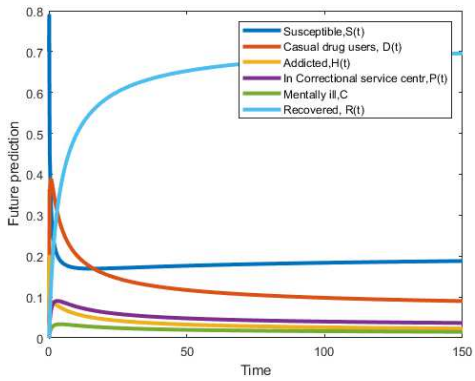
Figure 3: Results found using the two-step Lagrange scheme.

The stability analysis and numerical simulations reveal critical thresholds and long-term behaviors, particularly when examining different values of α and the influence of the function $g(x)$. These results underscore the importance of timely, dynamic interventions and demonstrate that control strategies sensitive to temporal changes can effectively reduce or eliminate illegal drug use. Moreover, the use of non-local operators in our model has proven instrumental in capturing complex phenomena that traditional methods might overlook. The simulation outcomes, which highlight the eventual dominance of the recovered group over time, shown from the simulation in Figs. 4(a), 4(b), 5(a), 5(b), 6(a), 6(b), 7(a), 7(b), further validate the model's applicability and predictive power.

Numerical approximations using three global derivative schemes – the two-step Lagrange, two-step Caputo-Fabrizio, and Midpoint Riemann-Liouville methods – provide additional insight into the system's behavior. While the model does have limitations and is not exhaustive, the results offer valuable perspectives on the dynamics of drug abuse and the potential impact of intervention strategies. Given the serious

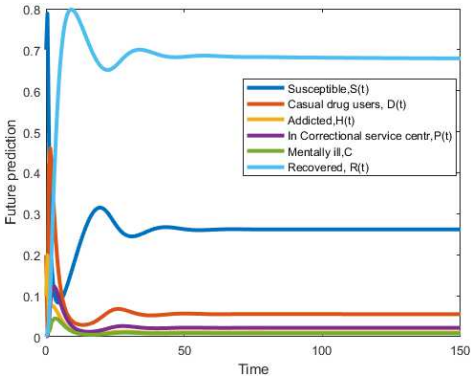


(a) $g(x) = 0.0001x$, $\alpha = 0.35$ for 150 days

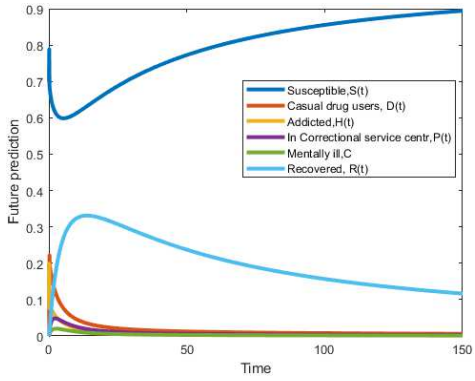


(b) $g(x) = 0.0001x$, $\alpha = 0.7$ for 150 days

Figure 4: Results found using Caputo-Fabrizio scheme.

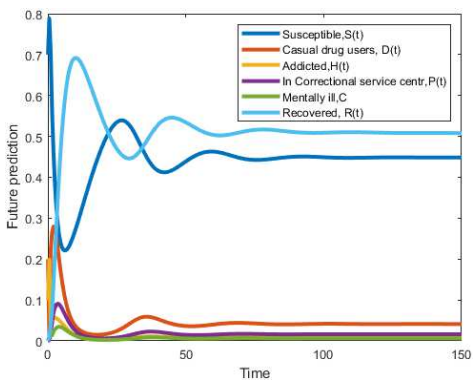


(a) $g(x) = x$, $\alpha = 0.35$ for 150 days

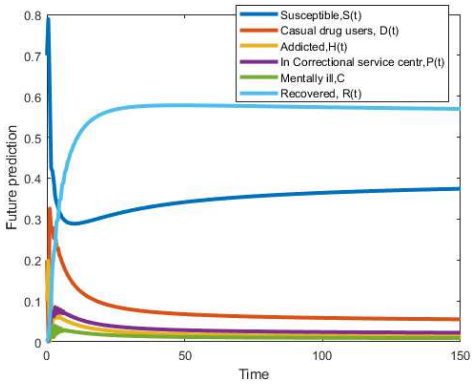


(b) $g(x) = x$, $\alpha = 0.7$ for 150 days

Figure 5: Results found using Caputo-Fabrizio scheme.

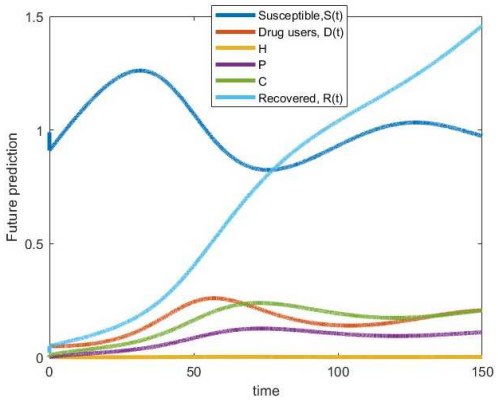


(a) $g(x) = 0.0001x$, $\alpha = 0.35$ for 150 days

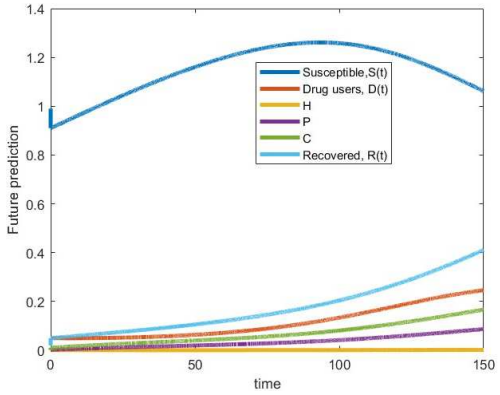


(b) $g(x) = 0.0001x$, $\alpha = 0.7$ for 150 days

Figure 6: Results found using midpoint Riemann-Liouville scheme.



(a) $g(x) = x$, $\alpha = 0.35$ for 150 days



(b) $g(x) = x$, $\alpha = 0.7$ for 150 days

Figure 7: Results found using midpoint Riemann-Liouville scheme.

health and social consequences associated with illicit drug use, these findings emphasize the need for continued research. Future studies should aim to refine the model, improve its accuracy, and explore additional intervention strategies to more effectively address the pervasive issue of illegal drug use in communities worldwide.

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