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Renormalized Solution of the Relativistic Boltzmann Equation in a Robertson-Walker Spacetime

D. S. Eyia Nnanga[1], R. Essono[2,*] and R. D. Ayissi[1]

[1] *Department of Mathematics, Faculty of Science, University of Yaounde I, BP 812, Yaounde, Cameroon*

[2] *Department of Mathematics, Faculty of Science, University of Ebolowa, BP 118, Ebolowa, Cameroon*

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Abstract. In this paper, the proof of the global existence of a renormalized or equivalently mild solution of the relativistic Boltzmann equation in a Robertson-Waker space-time is given for an initial value problem with initial data only satisfying the conditions of finite mass, energy, inertia and entropy.

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1 Introduction

We are concerned with global existence of solution of the initial value problem for the following dimensionless relativistic Boltzmann equation (RBE) in a Robertson-Walker space-time [6]

$$\partial_t f + \hat{p} \cdot \nabla_x f - 2\frac{\dot{G}}{G} p \cdot \nabla_p f = Q(f;f), \tag{1.1}$$

where different parts will be addressed below.

About the relativistic case, several authors have studied this problem by taking the Minkowski spacetime as background. Most of results available concern the study of mild solutions [1,3,8,9].

In their seminar paper DiPerna and Lions [1], based on new tools and techniques, have studied the non-relativistic Boltzmann equation. The key concept of their results is the notion of renormalized solution of the transport equation. By the velocity averaging, they permit to show the proof of global existence of weak solution via

*Corresponding author. *Emails:* `eyiannanga@gmail.com` (D. S. Eyia Nnanga), `essonorene@gmail.com` (R. Essono), `raoulayissi@yahoo.fr` (R. D. Ayissi)

compactness arguments. After this result, the desire to extend this method to the relativistic case becomes a problem. The response came firstly by Dudyński and Ekiel-Jeżewska in [2], in Minkowski spacetime, with an existence proof based on the causality property of the relativistic Boltzmann equation. By modifying the assumptions made on the scattering cross section in [2], more complicated in the relativistic case, Zhenglu [8,9] has given the proof of global existence of renormalized solution for the initial value problem for the relativistic Boltzmann equation using the DiPerna and Lions method's in Minkowski spacetime.

The objective of this paper is to use the same approach as in [1,9] and prove that there exists a global renormalized equivalently mild solution to the large data Cauchy problem for the relativistic Boltzmann in Robertson-Walker spacetime under the condition of initial data f_0 satisfying (2.32), that is

Theorem 1.1. *Let $K(g,\theta)$ be the relativistic collision kernel of the RBE* (2.14)*, and B_r a ball with center at the origin and radius r, $B(g) = \int_{S^2} K(g,\theta)d\Omega$. Assume that*

$$K(g,\theta) \geq 0, \quad a.e.\ in\ \ [0,+\infty)\times S^2, \quad K(g,\theta)\in L^1_{loc}(R^3\times S^2), \tag{1.2}$$

$$\frac{1}{(v^0)^2}\int_{B_r}\frac{B(g)}{v_1^0}dv_1 \ \ \rightarrow \ \ 0 \quad as \quad |v| \ \ \rightarrow \ \ +\infty, \quad \forall\, r,t\in(0,+\infty). \tag{1.3}$$

Then the RBE (1.1) *has a renormalized or equivalently a mild solution f through initial data f_0 with* (2.32) *satisfying the following properties:*

$$f \in C\big([0,+\infty); L^1(\mathbb{R}^3\times\mathbb{R}^3)\big) \tag{1.4}$$

$$L(f) \in L^\infty\big([0,+\infty); L^1\big(\mathbb{R}^3\times B_r\big)\big), \qquad \forall\, r\in(0,+\infty), \tag{1.5}$$

$$\frac{Q^+(f,f)}{1+f} \in L^\infty\big([0,+\infty); L^1\big(\mathbb{R}^3\times B_r\big)\big), \quad \forall\, r\in(0,+\infty), \tag{1.6}$$

$$\sup_{t\geq 0}\int\int_{\mathbb{R}^3\times\mathbb{R}^3} f(1+\ln f)dx\,dp < +\infty. \tag{1.7}$$

The challenge is the form of the Boltzmann equation in this spacetime. In order to use the DiPerna and Lions method, we base our approach in the transformation of Eq. (1.1) into a different equivalent form using covariant variables as in [5,6]. Then we follow the steps of [1,9]. In this work we use the similar assumptions on the scattering kernel already used in [9], namely

$$K(g,\theta) \geq 0 \quad \text{a.e.}, \tag{1.8}$$

$$z(1+z^2)K(z,\theta) \in L^1_{Loc}\big((0,+\infty)\times S^2\big), \tag{1.9}$$

$$\frac{1}{(v^0)^2}\int\int_{B_r\times S^2}\frac{gs^{\frac{1}{2}}K(g,\theta)}{v_1^0}d\Omega dv_1 \ \ \rightarrow \ \ 0, \quad |v| \ \ \rightarrow \ \ +\infty, \quad \forall\, r,t\in(0,+\infty), \tag{1.10}$$

where for $r>0$, B_r is a ball with its center in the origin and radius r, and where the other quantities will be specified in the sequel.

The plan of this paper is organized as follows. In Section 2, we transform the relativistic Boltzmann equation into another equivalent form, we give some a priori estimates satisfied by a solution of the initial value problem and present the main assumptions of this work. In Section 3, we give three formulations of solutions in the spirit of DiPerna and Lions paper [1]. In Section 4, is devoted to prove the existence of a renormalized solution.

2 Preliminaries

In this work Greek, indexes will be assumed to run from 0 to 3, while Latin indexes run from 1 to 3. We adopt the Einstein summation convention

$$a_\alpha b^\alpha = \sum_\alpha a_\alpha b^\alpha. \tag{2.1}$$

Notations $x^\alpha = (t, x)$ and $p^\alpha = (p^0, p)$ respectively represent a four-dimensional space-time dimensionless variable and a four-dimensional momentum-energy dimensionless variable. As background spacetime, the spatially flat Robertson-Walker spacetime where the metric tensor with signature $(-,+,+,+)$ can be written as

$$ds^2 = -dt^2 + R(t)^2[(dx^1)^2 + (dx^2)^2 + (dx^3)^2], \tag{2.2}$$

in which R is differentiable strictly positive increasing function of t called the scale factor. Using Christoffel symbols, computed in [6, 7], of the Levi-Cevita connection associated to the metric tensor, the relativistic Boltzmann equation in the Robertson-Walker spacetime can be written as

$$\partial_t f + \hat{p} \cdot \nabla_x f - 2\frac{\dot{G}}{G} p \cdot \nabla_p f = Q(f, f), \tag{2.3}$$

where $\hat{p}$ is defined by $\hat{p} = p/p^0$ with

$$p^0 = \sqrt{1 + G^2(t)|p|^2},$$

and

$$|p| = \left((p^1)^2 + (p^2)^2 + (p^3)^2\right)^{\frac{1}{2}}.$$

In (2.3), Q is a nonlinear operator called the collision operator which will be specified below. In instantaneous binary and elastic scheme, we consider that in a given position x, two particles of four-dimensional momenta $p^\alpha = (p^0, p)$ and $p_1^\alpha = (p_1^0, p_1)$ collide without destroying each other; the collision affecting only their momenta that change after collision. Let $p'^\alpha = (p'^0, p')$ and $p_1'^\alpha = (p_1'^0, p_1')$ be their four-dimensional momenta after the collision. By conservation of energy momenta, one has

$$p^\alpha + p_1^\alpha = p'^\alpha + p_1'^\alpha. \tag{2.4}$$

The collision operator is henceforth defined by

$$Q(f,g) = Q^{+}(f,g) - Q^{-}(f,g), \tag{2.5}$$

where

$$Q^{+}(f,g) = G^3(t)\int_{\mathbb{R}^3}\int_{S^2}\frac{g\sqrt{s}}{p^0q^0}K(g,\omega)f(t,x,p')f(t,x,p')g(t,x,q')d\omega dq, \tag{2.6}$$

$$Q^{-}(f,f) = G^3(t)\int_{\mathbb{R}^3}\int_{S^2}\frac{g\sqrt{s}}{p^0q^0}K(g,\omega)f(t,x,p)f(t,x,q)g(t,x,q)d\omega dq, \tag{2.7}$$

$$L(f) = G^3(t)\int_{\mathbb{R}^3}\int_{S^2}\frac{g\sqrt{s}}{p^0q^0}K(g,\omega)f(t,x,q)g(t,x,q)d\omega dq, \tag{2.8}$$

$$Q^{-}(f,f) = fL(f), \tag{2.9}$$

corresponding to the gain term and the loss term respectively. For simplicity, we sometimes abbreviate $f(t,x,p)$ by $f(p)$. In that expression of Q

- S^2 is the unit sphere of $\mathbb{R}^3$,
- s and g are called respectively the square of energy of the energy in the center of momentum system $p+q=0$ and the relative momentum. The are defined by

$$s = -(p_\alpha + q_\alpha)(p^\alpha + q^\alpha), \quad g = \sqrt{(p_\alpha + q_\alpha)(p^\alpha + q^\alpha)}, \tag{2.10}$$

- $K(g,\theta)$ is the differential cross-section or scattering kernel, it depends on the relative momentum g and the cross section θ is the scattering angle which is defined in $[0,\pi]$ by

$$\cos\theta = \frac{(p'_\alpha - q'_\alpha)(p^\alpha - p'^\alpha)}{g^2}. \tag{2.11}$$

2.1 Equation in new variables

In the remainder of this work, as in [6], we consider RBE (2.3) with covariant variables. So the distribution function f will be considered as a function of t, x and $p_k = g_{k\beta}p^\beta = R^2p^k$ with $k=1,2,3$. This change of variables was previously used in [5, 6]. In what follows, for simplicity we set $v=(v^1,v^2,v^3)$ where $v^k = G^2p^k$ and

$$v^0 = \sqrt{1+\frac{|v|^2}{G^2(t)}} = p^0.$$

In the sequel, we note

$$v_0^0 = \sqrt{1+\frac{|v|^2}{G^2(0)}}, \tag{2.12}$$

witch will be useful in many estimates.

With these new variables, we use $v'^k = G^2 p'^k$ and $v_1'^\alpha = G^2 p_1'^\alpha$ for the post collisional momenta. Now we write (2.3) in a new variables. Let $\tilde{f}(t,x,v) = f(t,x,p)$ then

$$\partial_t \tilde{f} = \partial_t f - 2\frac{\dot{G}}{G} p \cdot \nabla_p f, \quad \partial_x \tilde{f} = \partial_x f. \tag{2.13}$$

Straightforward computations lead to $dp = R^{-6} dv$. In the sequel by abuse of notation we will still write f instead of $\tilde{f}$. So with the new variables (2.3) is equivalent to

$$\partial_t f + \frac{1}{G^2} \hat{v} \cdot \partial_x f = Q(f,f), \tag{2.14}$$

where

$$\begin{aligned} Q(f,f) &= \frac{1}{G^3(t)} \int_{S^2} d\omega \int_{\mathbb{R}^3} du \frac{g\sqrt{s}}{v^0 u^0} K(\omega, g)[f(v')f(u') - f(v)f(u)] \\ &= Q^+(f,f)(t,x,v) - Q^-(f,f)(t,x,v), \end{aligned} \tag{2.15}$$

where $d\Omega = \sin\theta \, d\theta \, d\psi$, $0 \le \psi \le 2\pi$.

Since energy momentum and momenta are conserved quantities before and after collision of the two particles, one has

$$s = s', \quad g = g', \tag{2.16}$$

where

$$s' = |p' + p_1'|^2, \quad g' = \frac{1}{2}|p_1' - p'|, \tag{2.17}$$

so

$$\cos\theta = 1 + \frac{2a}{s-4}, \tag{2.18}$$

and $a = -|v - v'|^2$.

2.2 A priori estimates

If $Q(f;f)\varphi(v) \in L^1(\mathbb{R}^3)$ for any given $\varphi \in L^\infty(\mathbb{R}^3)$ and any given $f(v) \in L^1(\mathbb{R}^3)$, then we get that

$$\begin{aligned} \int_{\mathbb{R}^3} \varphi(v) Q(f,f)\, dv = \frac{1}{4G^3(t)} \int_{\mathbb{R}^3} dv \int_{S^2} d\omega \int_{\mathbb{R}^3} du \frac{g\sqrt{s}}{v^0 u^0} \\ \times K(\omega, g)[\varphi(v')\varphi(u') - \varphi(v)\varphi(u)]. \end{aligned} \tag{2.19}$$

If $\bar{\varphi} = \overline{b}_0 + b \cdot v + c_0 v^0$ where $\overline{b}_0 \in \mathbb{R}$, b, v, $v_0 \in \mathbb{R}^3$, $c_0 \in \mathbb{R}$, then

$$\int_{\mathbb{R}^3} \bar{\varphi}(v) Q(f,f)\, dv = 0.$$

This means that f satisfies the conservation laws of mass, momentum and energy if f is a distributional solution of RBE (2.14).

Next, let us show the property that entropy is always a nonincreasing function of t in the relativistic case. Multiplying (2.14) by $(1 + \ln f)$, integrating by parts over x

and v and using (2.19), we deduce at least formally, the following entropy relativistic identity:

$$\begin{aligned}&\frac{d}{dt}\int\int_{\mathbb{R}^3\times\mathbb{R}^3} f\log f\,dxdv\\&\quad+\frac{1}{G^3(t)}\int\int_{\mathbb{R}^3\times S^2} d\Omega dv_1\frac{gs^{\frac{1}{2}}K(g,\theta)}{v^0v_1^0}\\&\quad\times[f(v')f(u')-f(v)f(u)]\log\left(\frac{f(v')f(u')}{f(v)f(u)}\right)=0,\end{aligned}\tag{2.20}$$

or in other words

$$\frac{d}{dt}\int\int_{\mathbb{R}^3\times\mathbb{R}^3} f\log f\,dxdv+e(f)=0,\tag{2.21}$$

where

$$\begin{aligned}e\,(f)=&\frac{1}{2G^3(t)}\int\int_{\mathbb{R}^3\times S^2} d\Omega dv_1\frac{gs^{\frac{1}{2}}K(g,\theta)}{v^0v_1^0}\\&\times[f(v')f(u')-f(v)f(u)]\log\left(\frac{f(v')f(u')}{f(v)f(u)}\right)\geq 0.\end{aligned}\tag{2.22}$$

We deduce from (2.21) that $\int\int_{\mathbb{R}^3\times\mathbb{R}^3} f\log f dxdv$ is a nonincreasing function of time. Then the entropy of RBE defined by

$$H(t)=-\int\int_{\mathbb{R}^3\times\mathbb{R}^3} f\log f\,dxdv\tag{2.23}$$

is a nondecreasing function of time. In order to deduce a bound from (2.20), we need another estimate. This estimate is deduced from (2.14) by multiplying (2.14) by $v_0^0|x|^2$ and integrating by part over x and v, yields

$$\begin{aligned}&\frac{d}{dt}\int\int_{\mathbb{R}^3\times\mathbb{R}^3} v_0^0|x|^2 f dxdv\\&=2\int\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{v_0^0}{G^2(t)}fx\cdot\hat{v}dxdv\\&\leq\frac{2}{G^2(0)}\int\int_{\mathbb{R}^3\times\mathbb{R}^3} v_0^0 fx\cdot\hat{v}dxdv\\&\leq\frac{1}{G^2(0)}\int\int_{\mathbb{R}^3\times\mathbb{R}^3} v_0^0|x|^2 fdxdv+\frac{1}{G^2(0)}\int\int_{\mathbb{R}^3\times\mathbb{R}^3} v_0^0|v|^2 fdxdv.\end{aligned}\tag{2.24}$$

This leads to the following inequality:

$$\sup_{0\leq t\leq T}\int\int_{\mathbb{R}^3\times\mathbb{R}^3} v_0^0|x|^2 fdxdv\leq e^{\frac{T}{G^2(0)}}\int\int_{\mathbb{R}^3\times\mathbb{R}^3} f_0v_0^0\left(|x|^2+\frac{|v|^2}{G^2(0)}\right)dxdv\tag{2.25}$$

for any given $T>0$, multiplying (2.24) by $e^{-t/G^2(0)}$ and using the conservation of the mass.

The entropy can be controlled by the integral $\int\int_{\mathbb{R}^3\times\mathbb{R}^3} f|\ln f|dxdv$ for any non-negative solution of the RBE and so it is natural to estimate of this integral instead of entropy. So

$$\begin{aligned}
&\int\int_{\mathbb{R}^3\times\mathbb{R}^3} f|\log f|dxdv \\
&= \int\int_{\mathbb{R}^3\times\mathbb{R}^3} f\log f dxdv - 2\int\int_{f\le 1} f\log f dxdv \\
&\le \int\int_{\mathbb{R}^3\times\mathbb{R}^3} f\log f dxdv + 2\int\int_{\mathbb{R}^3\times\mathbb{R}^3} f\big(v_0^0(x-tv)^2+v_0^0\big)dxdv \\
&\quad + 2\int\int_{f<exp(v_0^0(x-tp)^2+v_0^0)} f\log\left(\frac{1}{f}\right)dxdv \\
&\le \int\int_{\mathbb{R}^3\times\mathbb{R}^3} f\log f dxdv + 2\int\int_{\mathbb{R}^3\times\mathbb{R}^3} f\big(v_0^0(x-tv)^2+v_0^0\big)dxdv + C_1,
\end{aligned} \tag{2.26}$$

where C_1 is some positive constant independent of f.

Using the fact that $\int\int_{\mathbb{R}^3\times\mathbb{R}^3} f\log f\, dxdv$ is a nonincreasing function of time, (2.25), (2.26) one deduces that

$$\sup_{0\le t\le T}\int\int_{\mathbb{R}^3\times\mathbb{R}^3} f|\log f|dxdv \le \int\int_{\mathbb{R}^3\times\mathbb{R}^3} f_0 U dxdv + C_1,$$

where

$$U = \log f_0 + 2v_0^0(T+1)e^{\frac{1}{G^2(0)}}|x|^2 + 2v_0^0\left(T^2+1+\frac{T+1}{G^2(0)}\right)|v|^2+2,$$

then the entropy can be estimated in any finite time T.

We consider the Cauchy problem

$$\begin{cases}\partial_t f + \dfrac{1}{G^2}\hat{v}\partial_x f = Q(f,f) & \text{in } [0,+\infty]\times\mathbb{R}^3\times\mathbb{R}^3,\\ f(0,x,v) = f_0(x,v) & \text{on } \mathbb{R}^3\times\mathbb{R}^3.\end{cases} \tag{2.27}$$

2.3 Main assumptions

In the present work, we suppose that:

Assumption 2.1. *About the scattering kernel bounds*

$$K(g,\theta)\ge 0 \quad a.e., \tag{2.28}$$

$$z(1+z^2)K(z,\theta)\in L^1_{Loc}\big((0,+\infty)\times S^2\big), \tag{2.29}$$

$$\frac{1}{(v^0)^2}\int\int_{B_r\times S^2}\frac{gs^{\frac{1}{2}}K(g,\theta)d\Omega dv_1}{v_1^0} \quad\to\quad 0,\quad |v|\to+\infty,\quad \forall\, r,t\in(0,+\infty). \tag{2.30}$$

Assumption 2.2. *About the scale factor G*

$$G(0) > 0, \quad G'(t) > 0. \tag{2.31}$$

Assumption 2.3. *About the initial condition of the Cauchy problem*

$$\begin{aligned} & f_0 \geq 0 \quad \text{a.e. in } \mathbb{R}^3 \times \mathbb{R}^3, \\ & \int\int_{\mathbb{R}^3\times\mathbb{R}^3} f_0\big(v_0^0 + v_0^0|x|^2 + v_0^0|v|^2 + \log f_0\big)dxdv < \infty. \end{aligned} \tag{2.32}$$

3 Formulation of solutions of RBE

Now we define three formulations of solution to the RBE useful below.

Definition 3.1. *A nonnegative function $f \in L^1{}_{loc}([0;+\infty[\times\mathbb{R}^3\times\mathbb{R}^3)$ satisfying RBE* (2.14) *in the distribution sense is called a distributional solution to RBE* (2.14).

Definition 3.2. *If a nonnegative function f belongs to $L^1{}_{loc}([0,+\infty[\times\mathbb{R}^3\times\mathbb{R}^3)$ and $u = \ln(1+f)$ solves*

$$\partial_t u + \frac{1}{G^2}\hat{v}\partial_x u = \frac{1}{1+f}Q(f,f)$$

in the distribution sense, where

$$\frac{1}{1+f}Q^{\pm}(f,f) \in L^1{}_{loc}([0,+\infty[\times\mathbb{R}^3\times\mathbb{R}^3),$$

then f is called a renormalized solution of RBE (2.14).

Definition 3.3. *A function f is called a mild solution of the RBE* (2.14) *if f is a nonnegative function which belongs to $L^1{}_{loc}([0;+\infty[\times\mathbb{R}^3\times\mathbb{R}^3)$ and for almost $(x,v) \in \mathbb{R}^3\times\mathbb{R}^3$,*

$$Q(f,f)^\sharp(t,x,v) \in L^1_{loc}([0,T[\times\mathbb{R}^3\times\mathbb{R}^3), \quad \forall\, T \in]0,+\infty[,$$

and

$$f^\sharp(t,x,v) - f^\sharp(s,x,v) = \int_s^t Q(f,f)^\sharp(\mu,x,v)d\mu, \quad \forall\, 0 \leq s < t < +\infty,$$

where $h^\sharp$ denotes for any measurable function h on $[0;+\infty[\times\mathbb{R}^3\times\mathbb{R}^3$ the following restriction to characteristics:

$$h^\sharp(t,x,v) = h\big(t,X(t,x,v),v\big)$$

with

$$X(t,x,v) = x + \left(\int_0^t \frac{1}{G^2(\tau)\sqrt{1+|v|^2G^{-2}(\tau)}}d\tau\right)v. \tag{3.1}$$

Using the above definitions, and using DiPerna and Lions [1], we can show that the following Theorems 3.1-3.8 hold. They show the relation between these solution formulations.

Theorem 3.1. *A function f is a renormalized solution of RBE if f is a distributional solution of RBE and $Q^{\pm}(f,f) \in L^1{}_{loc}([0,+\infty[\times\mathbb{R}^3\times\mathbb{R}^3)$.*

Theorem 3.2. *If a function f is a renormalized solution of the RBE, then the composition $\beta(f)$ is a distributional solution of*

$$\partial_t\beta + \frac{1}{G^2}\partial_x\beta = \beta'(f)Q(f,f)$$

for all $\beta = \beta(t) \in C^1([0,+\infty[)$ such that $|\beta'(t)| \leq C/(1+t)$ for some positive constant C.

Theorem 3.3. *A function f is a distributional solution of RBE if f is a renormalized solution of RBE and $Q^{\pm}(f,f) \in L^1{}_{loc}([0,+\infty[\times\mathbb{R}^3\times\mathbb{R}^3)$.*

Theorem 3.4. *If $Q^{\mp}(f,f)(t,x,v) \in L^1_{loc}([0,+\infty[\times\mathbb{R}^3\times\mathbb{R}^3)$ and f is a distributional solution of RBE , then f is a mild solution of RBE.*

Theorem 3.5. *If $Q^{\mp}(f,f)(t,x,v) \in L^1_{loc}([0,+\infty[\times\mathbb{R}^3\times\mathbb{R}^3)$, $f^\sharp$ is absolutely continuous with respect to t for almost $(x,v) \in \mathbb{R}^3\times\mathbb{R}^3$ and f is a mild solution of RBE, then f is a distributional solution of RBE.*

Theorem 3.6. *If $f$is a renormalized solution of RBE and $f^\sharp$ is absolutely continuous with respect to t for almost $(x,v) \in \mathbb{R}^3\times\mathbb{R}^3$, then f is a mild solution of RBE.*

Theorem 3.7. *Suppose $Q^{\mp}(f,f)/(1+f) \in L^1_{loc}([0,+\infty[\times\mathbb{R}^3\times\mathbb{R}^3)$ and f is a mild solution of RBE, then f is a renormalized solution of RBE.*

Theorem 3.8. *Suppose $L(f) \in L^1_{loc}(]0,T[\times\mathbb{R}^3\times B_r)$ for all $0<r$, $T<+\infty$. Let us write*

$$F^\sharp(t,x,v) = \int_0^t L^\sharp(f)(\tau,x,v)d\tau.$$

Then f is a mild solution of RBE if and only if for almost $(x,v)\in\mathbb{R}^3\times\mathbb{R}^3$ and $0\leq s<t<+\infty$ the following exponential multiplier form holds:

$$\begin{aligned} &f^\sharp(t,x,v) - f^\sharp(s,x,v)\exp\left[-\left(F^\sharp(t)-F^\sharp(s)\right)\right] \\ &= \int_s^t Q^+(f,f)^\sharp(\tau,x,v)\exp\left[-\left(F^\sharp(t)-F^\sharp(s)\right)\right]d\tau. \end{aligned}$$

In order to read this work, it is recommended to be aware of the methods used in [1, 9]. The core of this method is based in two particular points: some hypothesis and the mean average result as the one used in [1, 9]. It is important to remain this aspect, so that one should not reproduce many results which can be found in [1, 9]. Firstly, one notices that hypothesis (2.28)-(2.30), (2.32) are similar to the one adopted in [9]

$$B(g,\theta) \geq 0 \quad \text{a.e.} \quad z(1+z^2)B(z,\theta) \in L^1_{Loc}\big((0,+\infty)\times S^2\big), \tag{3.2}$$

$$\frac{1}{(p^0)^2}\int_{B_R}\frac{A(g)}{p_1^0}d^3p_1 \;\to\; 0 \quad \text{as} \quad |p| \;\to\; +\infty, \quad \forall R \in (0,\infty). \tag{3.3}$$

About the mean average, the following result proved in [4] will be useful.

Lemma 3.1. *Define the operator* T *from* $L^1(dx \otimes d\mu(v))$ *by* $Tf = \int u(x,v)\, d\mu(v)$, *where* u *is the unique solution in* $L^1(dx \otimes d\mu(v))$ *on the transport equation*

$$u + v \cdot \partial_x u = f, \quad x \in \mathbb{R}^N, \quad v \in \mathbb{R}^N.$$

If $K \subset L^1(dx \otimes d\mu(v))$ *is bounded and uniformly integrable, then* $T(K)$ *is compact in* $L^1_{loc}(dx)$.

But for the use of Lemma 3.1, we need to prove that $f_n + \tilde{Q}_n(f_n, f_n)$ is uniformly integrable where f_n is the unique solution of the approximation RBE (4.2). But this is a classical result according to (2.30)-(2.32) and the proof made in [4, pp. 148-152].

4 Proof of Theorem 1.1

The proof of Theorem 1.1 depends on the same approximation scheme given by DiPerna and Lions [1]. Initially the collision kernel is truncated to obtain $K_n(g,\theta) \in L^\infty \cap L^1(\mathbb{R}^3, L^1(S^2))$ such that

$$\int\int_{B_r \times S^2} |K_n(g,\theta) - K(g,\theta)| dv d\Omega \;\; \to \;\; 0 \tag{4.1}$$

uniformly in $\{v_1 : |v_1| \le k\}$ as $n \to \infty$ for all $r \in (0,+\infty)$. Then follows the resolution of the approximation equation

$$\frac{\partial f^n}{\partial t} + \frac{1}{G^2(t)}\hat{v} = \tilde{Q}_n(f_n, f_n) \quad \text{in } (0,+\infty) \times \mathbb{R}^3 \times \mathbb{R}^3 \quad \text{with} \quad f^n|_{t=0} = f_0^n, \tag{4.2}$$

where

$$Q_n(f,f) = \frac{1}{G^3(t)} \int_{S^2} d\omega \int_{\mathbb{R}^3} du \frac{g\sqrt{s}}{v^0 u^0} K_n(\omega, g)[f(v')f(u') - f(v)f(u)], \tag{4.3}$$

$$\tilde{Q}_n(f,f) = \left(1 + \frac{1}{n}\int_{\mathbb{R}^3} f dv\right)^{-1} Q_n(f_n, f_n). \tag{4.4}$$

On deduces from (4.3) and (2.31)

$$\|\tilde{Q}_n(\varphi,\varphi)\|_{L^\infty([0,+\infty)\times\mathbb{R}^3\times\mathbb{R}^3)} \le C_n \|\varphi\|_{L^\infty([0,+\infty)\times\mathbb{R}^3\times\mathbb{R}^3)}, \tag{4.5}$$

$$\|\tilde{Q}_n(\varphi,\varphi)\|_{L^1(\mathbb{R}^3\times\mathbb{R}^3)} \le C_n \|\varphi\|_{L^1(\mathbb{R}^3\times\mathbb{R}^3)}, \tag{4.6}$$

$$\|\tilde{Q}_n(\varphi,\varphi) - \tilde{Q}_n(\psi,\psi)\|_{L^1(\mathbb{R}^3\times\mathbb{R}^3)} \le C_n \|\varphi - \psi\|_{L^1(\mathbb{R}^3\times\mathbb{R}^3)}, \tag{4.7}$$

where every where C_n is nonnegative constant independent of φ and ψ.

By following DiPerna and Lions [1], the initial data f_0 can be first truncated and regularized to get a sequence of nonnegative functions (f_0^n) is an approximation sequence of f_0 obtained such that we first approximate the initial condition f_0 by truncating it and regularizing the truncated function by convolution to obtain $\tilde{f}_0^n \in \mathcal{D}(\mathbb{R}^3 \times \mathbb{R}^3)$ such that $\tilde{f}_0^n \geq 0$ and

$$\int\int dxdv|f_0 - \tilde{f}_0^n|(1+|x|^2+\tilde{v}^0+|v|^2) \underset{n}{\rightarrow} 0, \tag{4.8}$$

$$\int\int dxdv\tilde{f}_0^n|\ln \tilde{f}_0^n| \leq C \quad \text{independent of } n. \tag{4.9}$$

Then we set

$$f_0^n = \tilde{f}_0^n + \frac{1}{n}\exp -\frac{1}{2}(|x|^2+|v|^2+\tilde{v}^0).$$

We deduce that (4.8) and (4.9) hold with $\tilde{f}_0^n$ replaced by f_0^n.

5 Conclusion

In this study, we have provided a construction of renormalized solution to the relativistic Boltzmann equation in a Robertson-Walker space time. This study extends the approach proposed in [1] to a different metric of space-time. We have been obliged to consider different assumptions in order to produce this result and dominate the mathematical challenge created by this procedure. It is proved that renormalized solution is equivalent to mild solution, therefore this study gives a different approach to obtain the result given in [9] in a different spacetime background.

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