African Journal for Industrial and Applied Mathematics
Afr. J. Ind. Appl. Math., Vol. **xx**, No. x, pp. 1-10

DOI: 10.4208/ajiam.2025-0008
xxx 2025

Existence and Stability of Solutions to Differential Equations via the Deformable Derivative and Laplace Transform

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Received 11 December 2024; Accepted (in revised version) 5 April 2025

Abstract. In this short note, we establish the Existence and stability of solutions of an abstract semilinear differential equation governed by the so-called deformable derivative. We achieve our results using Banach's contraction principle, the Laplace transform, and the Gronwall inequality. These results are new in the context of fractional differential equations.

AMS subject classifications: 26A33, 26D1, 47H10
Key words: Deformable derivative, Fractional differential equations, Banach's contraction principle.

1 Introduction

Increasing research is being conducted on fractional calculus, and results are being obtained of late due to the success in using fractional differential equations in depicting and modeling natural and real-world phenomena better than the usual differential equations. Authors F. Zulfeqarr, A. Ujlayan, and P. Ahuja introduced a fractional derivative, the deformable derivative, in 2017 (see [31]). Unlike the most known fractional derivatives such as Riemann-Liouville and Caputo, deformable derivative doesn't involve an integral. Instead, it uses the limit approach as in the usual derivative. This derivative and the usual derivative are linearly related. Recently, an increasing number researchers are exploring this derivative, and the reader may refer to manuscripts [11–13, 24–26, 31] for an in-depth reading. In [11], we established the Existence and uniqueness of solutions to impulsive Cauchy problems involving the

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deformable derivative with local and nonlocal conditions. In [12, 13], we established the Existence, uniqueness, and stability of a solution for a fractional differential equation with delay and infinite delay, respectively. This paper is motivated by [17, 19], and we solve problem (1)-(2) in Section 3 using Laplace transform, using tools such as the Mittag-Leffler function and the Gronwall inequality.

2 Preliminaries

Definition 2.1 ([31]). *Let f be a real valued function on [a,b]. For $0 \leq \alpha \leq 1$, the Deformable derivative of f of order α at $t \in (a, b)$ is defined as:*

$$D^\alpha f(t) = \lim_{\epsilon \to 0} \frac{(1+\epsilon\beta) f(t+\epsilon\alpha) - f(t)}{\epsilon},$$

where $\alpha + \beta = 1$.We say that f is α-differentiable at t, provided the limit exists.

Remark 2.1. *If $\alpha = 1$, then $\beta = 0$, we recover the usual derivative. This shows that the deformable derivative is more general than the usual derivative.*

Definition 2.2 ([31]). *For f defined on $[a, b], \alpha \in (0, 1]$, the α-integral of the function f is defined by*

$$I_a^\alpha f(t) = \frac{1}{\alpha} e^{\frac{-\beta}{\alpha} t} \int_a^t e^{\frac{\beta}{\alpha} x} f(x) dx, \qquad t \in [a, b],$$

where $\alpha + \beta = 1$. When $a = 0$ we use the notation

$$I^\alpha f(t) = \frac{1}{\alpha} e^{\frac{-\beta}{\alpha} t} \int_0^t e^{\frac{\beta}{\alpha} x} f(x) dx.$$

Remark 2.2. *If $\alpha = 1$, then $\beta = 0$, we recover the usual Riemann integral. This also shows that the α-integral is more general than the usual Riemann integral.*

Theorem 2.1 ([31]). *A differentiable function f at a point $t \in (a, b)$ is always α-differentiable at that point for any α, and we have the following equation:*

$$D^\alpha f(t) = \beta f(t) + \alpha D f(t).$$

Theorem 2.2 ([25, 31]). *The operators D^α and I_a^α possess the following properties:*
Let $\alpha, \alpha_1, \alpha_2 \in (0, 1]$ such that $\alpha + \beta = 1$, $\alpha_i + \beta_i = 1$ for $i = 1, 2$.

(i) Let f be differentiable at a point t for some α. Then it is continuous there.

(ii) Suppose f and g are α-differentiable. Then

$$\begin{aligned} D^\alpha (f \circ g)(t) &= \beta (f \circ g)(t) + \alpha D (f \circ g)(t) \\ &= \beta (f \circ g)(t) + \alpha f'(g(t)) g'(t). \end{aligned}$$

(iii) Let f be continuous on [a,b]. Then $I_a^\alpha f$ is α- differentiable in (a,b), and we have

$$D^\alpha(I_a^\alpha f(t)) = f(t),$$
$$I_a^\alpha(D^\alpha f(t)) = f(t) - e^{\frac{\beta}{\alpha}(a-t)} f(a).$$

(iv) $D^\alpha\left(\dfrac{f}{g}\right) = \dfrac{gD^\alpha(f) - \alpha f Dg}{g^2}$.
(v) Linearity : $D^\alpha(af + bg) = aD^\alpha f + bD^\alpha g$.
(vi) Commutativity : $D^{\alpha_1} \cdot D^{\alpha_2} f = D^{\alpha_2} \cdot D^{\alpha_1} f$.
(vii) For a constant c, $D^\alpha(c) = \beta c$.
(viii) $D^\alpha(fg) = (D^\alpha f)g + \alpha f Dg$.
(ix) Linearity : $I_a^\alpha(bf + cg) = bI_a^\alpha f + cI_a^\alpha g$.
(x) Commutativity : $I_a^{\alpha_1} I_a^{\alpha_2} f = I_a^{\alpha_2} I_a^{\alpha_1} f$.

Corollary 2.1 ([31]). *An α-differentiable function f defined in (a, b) is differentiable in that interval.*

Lemma 2.1 ([9]). *If*

$$x(t) \leq h(t) + \int_{t_0}^{t} k(s)x(s)ds, \qquad t \in [t_0, T),$$

where all functions involved are continuous on $[t_0, T), T \leq +\infty$, and $k(t) \geq 0$, then $x(t)$ satisfies

$$x(t) \leq h(t) + \int_{t_0}^{t} h(s)k(s)\, exp\left[\int_{s}^{t} k(\omega)d(\omega)\right] ds, \qquad t \in [t_0, T).$$

If $x(t)$ is non-decreasing, then

$$x(t) \leq h(t)\, exp\left(\int_{0}^{t} k(s)ds\right), \qquad t \in [t_0, T).$$

Definition 2.3 ([20]). *Let a function $f(t)$ be piecewise continuous for $t \in (0, \infty)$, and be of exponential order. Then its Laplace transform is defined as*

$$\mathcal{L}[f(t)] = F(s) = \int_{0}^{\infty} e^{-st} f(t)dt.$$

Definition 2.4 ([1]). *Let $f : [0, \infty) \to \mathbb{R}$. For $\alpha \in (0, 1]$, the deformable Laplace transform of f, $\mathcal{L}^\alpha f$, is given by the integral:*

$$\mathcal{L}^\alpha f(t) = F^\alpha(p) = \int_{0}^{\infty} e^{-st} f(t) e_\alpha(t, 0) d_\alpha t,$$

where

$$e_\alpha(t, 0) = e^{-\frac{1}{\alpha}\int_0^t \beta du}, \ \alpha + \beta = 1, \ p = s + \frac{\beta}{\alpha}, \ d_\alpha t = \frac{1}{\alpha} dt.$$

Definition 2.5 ([3]). *Let $(\mathcal{C}, d)$ be a metric space. A mapping $T : \mathcal{C} \to \mathcal{C}$ is called a contraction on $\mathcal{C}$ if there exists a constant $\gamma \in (0,1)$ such that $d(T(x), T(y)) \leq \gamma d(x,y)$ $\forall x, y \in \mathcal{C}$.*

Theorem 2.3 ([3]). *Let $(\mathcal{C}, d)$ be a complete metric space. Then, every contraction has a unique fixed point.*

Lemma 2.2 ([16]). *Suppose $b(t)$ is nonnegative and locally integrable on $0 \leq t < T$ (some $T \leq +\infty$), and $a(t)$ is nonnegative and locally integrable on $0 \leq t < T$ with*

$$a(t) \leq b(t) + p\int_0^t (t-s)^{\alpha-1} a(s)ds.$$

Then

$$a(t) \leq b(t) + \theta \int_0^t \mathbb{E}'_\phi[\theta(t-s)]a(s)ds, \qquad 0 \leq t < T,$$

where

$$\theta = \left(p\,\Gamma(\phi)\right)^{\frac{1}{\phi}}, \quad E_\phi(z) = \sum_{n=0}^{\infty} \frac{z^{n\phi}}{\Gamma(n\phi+1)}, \quad E'_\phi(z) = \frac{d}{dz}E_\phi(z),$$

$$E'_\phi(z) \simeq \frac{z^{\phi-1}}{\Gamma(\phi)} \quad \text{as } z \to 0^+, \qquad E'_\phi(z) \simeq \frac{1}{\phi}e^z \quad \text{as } z \to +\infty,$$

$$E_\phi(z) \simeq \frac{1}{\phi}e^z \quad \text{as } z \to +\infty.$$

If $b(t) \equiv b$, then $a(t) \leq b\mathbb{E}_\phi(\theta t)$.

3 Main results

In this section,we establish the Existence and uniqueness of the solution for

$$D^\alpha x(t) = Ax(t) + f(t), \qquad t \in \mathcal{J} \tag{3.1}$$

$$x(0) = \varrho, \tag{3.2}$$

where D^α is the deformable derivative operator, A is an $n \times n$ matrix, $0 < \alpha < 1$, $f(t)$ is a vector function, which is the forcing term, ϱ is a real constant and $\mathcal{J} = [0, T]$.

Let $BC(\mathcal{J}, \mathbb{R}^n)$ be the space of all continuous and bounded functions $\mathcal{J} \to \mathbb{R}^n$ equipped with the sup-norm $\|x\|_\infty = \sup_{t\in\mathcal{J}} \|x(t)\|$. We now derive a formula of variation of constants as follows:

Theorem 3.1. *The system* (1)-(2) *is equivalent to the following integral equation*

$$x(t) = \varrho e^{\frac{-\beta}{\alpha}t} + \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}[Ax(\tau) + f(\tau)]d\tau, \qquad t \in \mathcal{J}. \tag{3.3}$$

Proof. Assume (1)-(2). Taking the α-integral of (1) according to definition 2.2, and applying the inverse property in Theorem 2.2, part (iii), we obtain the Existence of a solution for Eqs.(1)-(2).

Conversely, assuming (3) and taking the deformable derivative (D^{α}) of both sides of the equation and using Theorem 2.1, we get (1).

Definition 3.1. *A function $x \in BC(\mathcal{J}, \mathbb{R}^n)$ is said to be a mild solution of* (1)-(2) *if*

$$x(t) = \varrho e^{\frac{-\beta}{\alpha}t} + \frac{1}{\alpha} e^{\frac{-\beta}{\alpha}t} \int_0^t e^{\frac{\beta}{\alpha}s}[Ax(\tau) + f(\tau)]d\tau, \qquad t \in \mathcal{J},$$

provided the integral exists.

We denote $\|A\| := \max_{1 \le i,j \le n} |a_{i,j}|$. Now we will use Banach's contraction principle to prove the following theorem. In what follows, we assume that $\mathcal{J} = [0, T]$.

Theorem 3.2. *Let $f : \mathcal{J} \to \mathbb{R}^n$ be a smooth function. Assume $\frac{||A||}{\beta} < 1$. Then, Problem* (1)-(2) *has a unique solution.*

Proof. Let $x, y \in BC(\mathcal{J}, \mathbb{R}^n)$. Define an operator $\Omega : BC(\mathcal{J}, \mathbb{R}^n) \to BC(\mathcal{J}, \mathbb{R}^n)$ as

$$\Omega x(t) = e^{\frac{-\beta}{\alpha}t} \left[\varrho + \frac{1}{\alpha} \int_0^t e^{\frac{\beta}{\alpha}\tau}[Ax(\tau) + f(\tau)]d\tau \right].$$

Then

$$\begin{aligned} ||\Omega x(t) - \Omega y(t)|| &= \left\| e^{\frac{-\beta}{\alpha}t} \frac{1}{\alpha} \int_0^t e^{\frac{\beta}{\alpha}\tau} A[x(t) - y(t)]d\tau \right\| \\ &\le e^{\frac{-\beta}{\alpha}t} \frac{\|A\|}{\alpha} \int_0^t e^{\frac{\beta}{\alpha}\tau} \|x(t) - y(t)\| d\tau \\ &\le \|A\| \|x - y\|_\infty e^{\frac{-\beta}{\alpha}t} \frac{1}{\alpha} \int_0^t e^{\frac{\beta}{\alpha}\tau} d\tau \\ &\le \|A\| \|x - y\|_\infty \frac{e^{\frac{-\beta}{\alpha}t}}{\beta} \left[e^{\frac{\beta}{\alpha}t} - 1 \right] \\ &\le \|A\| \|x - y\|_\infty \frac{[1 - e^{\frac{-\beta}{\alpha}t}]}{\beta} \\ &\le \frac{\|A\|}{\beta} \|x - y\|_\infty. \end{aligned}$$

Therefore

$$||\Omega x - \Omega y||_\infty \le \frac{||A||}{\beta} ||x - y||_\infty.$$

Since $\frac{\|A\|}{\beta} < 1$, Ω is a contraction and thus it has a unique fixed point, which is the unique solution to Problem (1)-(2).

4 Semilinear case

Consider the following system

$$D^{\alpha}x(t) = Ax(t) + g(t, x(t)), \qquad t \in \mathcal{J} \tag{4.1}$$
$$x(0) = \varrho. \tag{4.2}$$

Theorem 4.1. *The system* (4)-(5) *is equivalent to the following integral equation*

$$x(t) = \varrho e^{\frac{-\beta}{\alpha}t} + \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}[Ax(\tau) + g(\tau, x(\tau))]d\tau, \qquad t \in \mathcal{J}. \tag{4.3}$$

Proof. The proof is similar to the proof for Theorem 3.1.

Definition 4.1. *A function $x \in BC(\mathcal{J}, \mathbb{R}^n)$ is said to be a mild solution of* (4)-(5) *if*

$$x(t) = \varrho e^{\frac{-\beta}{\alpha}t} + \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}[Ax(\tau) + g(\tau, x(\tau))]d\tau, \qquad t \in \mathcal{J},$$

provided the integral exists.

Now let's make the following assumptions:

(H1) $g : \mathcal{J} \times \mathbb{R}^n \to \mathbb{R}^n$ is of Carathéodory.

(H2) There exists a positive number γ such that $\|g(t,x) - g(t,y)\| \leq \gamma\|x - y\|, \ \forall t \in \mathcal{J}$ and $x, y \in \mathbb{R}^n$.

(H2') $\|g(t,x) - g(t,y)\| \leq L_g(t)\|x - y\|, t \in \mathbb{R}^+$, where $L_g \in L^1(\mathbb{R}^+) \cap BC(\mathbb{R}^+)$.

Theorem 4.2. *Under assumptions* (H1)-(H2)*, if $\frac{(\|A\|+\gamma)}{\beta} < 1$, the Cauchy problem* (4)-(5) *has a unique mild solution.*

Proof. Define $G : BC(\mathcal{J}, \mathbb{R}^n) \to BC(\mathcal{J}, \mathbb{R}^n)$ by

$$Gx(t) = e^{\frac{-\beta}{\alpha}t}\left[\varrho + \frac{1}{\alpha}\int_0^t e^{\frac{\beta}{\alpha}\tau}[Ax(\tau) + g(\tau, x(\tau))]d\tau\right].$$

Then

$$\begin{aligned}
&\|Gx(t) - Gy(t)\| \\
&\quad = \left\| e^{\frac{-\beta}{\alpha}t}\frac{1}{\alpha}\int_0^t e^{\frac{\beta}{\alpha}\tau}A[x(\tau) - y(\tau)]d\tau + e^{\frac{-\beta}{\alpha}t}\frac{1}{\alpha}\int_0^t e^{\frac{\beta}{\alpha}\tau}[g(\tau, x(\tau)) - g(\tau, y(\tau))]\right\| \\
&\quad \leq e^{\frac{-\beta}{\alpha}t}\frac{1}{\alpha}\int_0^t e^{\frac{\beta}{\alpha}\tau}\|A[x(\tau) - y(\tau)]\|d\tau + e^{\frac{-\beta}{\alpha}t}\frac{1}{\alpha}\int_0^t e^{\frac{\beta}{\alpha}\tau}\|[g(\tau, x(\tau)) - g(\tau, y(\tau))]\| \\
&\quad \leq e^{\frac{-\beta}{\alpha}t}\frac{1}{\alpha}\|A\|\|x - y\|_\infty \int_0^t e^{\frac{\beta}{\alpha}\tau}d\tau + e^{\frac{-\beta}{\alpha}t}\frac{1}{\alpha}\gamma\|x - y\|_\infty \int_0^t e^{\frac{\beta}{\alpha}\tau}d\tau \\
&\quad \leq \|A\|\|x - y\|_\infty \frac{[1 - e^{\frac{-\beta}{\alpha}t}]}{\beta} + \gamma\|x - y\|_\infty \frac{[1 - e^{\frac{-\beta}{\alpha}t}]}{\beta} \\
&\quad \leq \frac{(\|A\| + \gamma)}{\beta}\|x - y\|_\infty.
\end{aligned}$$

Thus
$$\|Gx - Gy\|_\infty \le \frac{(\|A\| + \gamma)}{\beta}\|x - y\|_\infty.$$

Therefore, G has a unique fixed point, which is the solution of the Cauchy problem. The proof is complete.

Theorem 4.3. *Under assumptions* (H1) *and* (H2′), *problem* (4)-(5) *has a unique mild solution.*

Proof. We use the operator G that was defined in Theorem 4.2. Let $x_1, x_2 \in BC(\mathcal{J}, \mathbb{R}^n)$. Then

$$\begin{aligned}
&\|(Gx_1)(t) - (Gx_2)(t)\| \\
&\quad = \left\| \frac{1}{\alpha} e^{\frac{-\beta}{\alpha}t} \int_0^t e^{\frac{\beta}{\alpha}\tau} A[x_1(\tau) - x_2(\tau)]d\tau + \frac{1}{\alpha} e^{\frac{-\beta}{\alpha}t} \int_0^t e^{\frac{\beta}{\alpha}\tau} [g(\tau, x_1(\tau)) - g(\tau, x_2(\tau))] \right\| \\
&\quad \le \frac{1}{\alpha} e^{\frac{-\beta}{\alpha}t} \int_0^t e^{\frac{\beta}{\alpha}\tau} \|A[x_1(\tau) - x_2(\tau)]\|d\tau + \frac{1}{\alpha} e^{\frac{-\beta}{\alpha}t} \int_0^t e^{\frac{\beta}{\alpha}\tau} \|[g(\tau, x_1(\tau)) - g(\tau, x_2(\tau))]\|d\tau. \\
&\quad \le \frac{\|A\|}{\alpha} e^{\frac{-\beta}{\alpha}t} \int_0^t e^{\frac{\beta}{\alpha}\tau} \|x_1(\tau) - x_2(\tau)\|d\tau + \frac{1}{\alpha} e^{\frac{-\beta}{\alpha}t} \int_0^t e^{\frac{\beta}{\alpha}\tau} L_g(\tau)\|x_1(\tau)) - x_2(\tau))\|d\tau. \\
&\quad \le \frac{\|A\|}{\beta} e^{\frac{\beta}{\alpha}T} \int_0^t \|x_1(\tau) - x_2(\tau)\|d\tau + \frac{1}{\beta} e^{\frac{\beta}{\alpha}T} \int_0^t e^{\frac{\beta}{\alpha}\tau} L_g(\tau)\|x_1(\tau)) - x_2(\tau))\|d\tau. \\
&\quad \le \|A\|M \int_0^t \|x_1(\tau) - x_2(\tau)\|d\tau + M \int_0^t L_g(\tau)\|x_1(\tau)) - x_2(\tau))\|d\tau,
\end{aligned}$$

letting $\frac{1}{\beta} e^{\frac{\beta}{\alpha}T} = M$.
Repeating the argument, we get

$$\begin{aligned}
&\|(G^n x_1)(t) - (G^n x_2)(t)\| \\
&\quad \le (\|A\|M)^n \int_0^t \int_0^\tau \cdots \int_0^{\tau_{n-2}} \|x_1(\tau_{n-1}) - x_2(\tau_{n-1})\|d\tau_{n-1} \cdots d\tau_1 d\tau \\
&\qquad + M^n \int_0^t \int_0^\tau \cdots \int_0^{\tau_{n-2}} L_g(\tau)L_g(\tau_1) \cdots L_g(\tau_1)\|x_1(\tau_{n-1}) - x_2(\tau_{n-1})\|d\tau_{n-1} \cdots d\tau_1 d\tau \\
&\quad \le \frac{(M\|A\|)^n}{n!}\|x_1 - x_2\|_\infty + \frac{M^n}{n!}\left(\int_0^t L_g(\tau)d\tau\right)^n \|x_1 - x_2\|_\infty \\
&\quad \le \frac{1}{n!}\left[(M\|A\|)^n + (M\|L_g\|_1)^n\right]\|x_1 - x_2\|_\infty.
\end{aligned}$$

For sufficiently large n, $\frac{1}{n!}\left[(M\|A\|)^n + (M\|L_g\|_1)^n\right] < 1$ and by Banach's contraction principle, there is a fixed point, which is the unique solution.

Example 1. Assume that for $0 < t_1 < t_2 < 1$, there exist real numbers $0 < Q < L < 1$ so that $|f(t_1) - f(t_2)| < Q|t_1 - t_2| < L\|x - y\|,\ x, y \in R^n$.
Consider the following linear equation:

$$f(t) = \frac{e^{3t}}{80 + e^{3t}}, \qquad t \in [0,1]$$

$$\begin{aligned} |f(t_1) - f(t_2)| &= \left| \frac{e^{3t_1}}{80 + e^{3t_1}} - \frac{e^{3t_2}}{80 + e^{3t_2}} \right| \\ &\leq \frac{1}{4}|t_1 - t_2| \\ &\leq \frac{1}{3}\|x - y\| \end{aligned}$$

Thus, Theorem 3.2 is satisfied.
Example 2. Consider Eqs.(4)-(5). Let

$$g(t, x(t)) = \frac{e^t}{e^t + 10} x(t)^2, \qquad t \in [0,1], x, y \in R^n \ \ and \ \ \|x + y\| \leq 1.$$

Then

$$\begin{aligned} \|g(t,x) - g(t,y)\| &= \left\| \frac{e^t}{e^t + 10} x(t)^2 - \frac{e^t}{e^t + 10} y(t)^2 \right\| \\ &= \frac{e^t}{e^t + 10} \left\| x(t)^2 - y(t)^2 \right\| \\ &\leq \frac{e^t}{e^t + 10} \|x - y\| \\ &\leq \frac{1}{2}\|x - y\|. \end{aligned}$$

Thus, (H2) is satisfied with $\gamma = \frac{1}{2}$. Under assumptions (H1)-(H2), and applying Theorem 4.2, Cauchy problem (4)-(5) has a unique solution in $\mathcal{J}$.

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