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Separation of Sequences and Multipliers in the Space of Tempered Distributions

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Abstract. We consider the notions of ν-separation and (N,ν)-separation for increasing sequences that tend to infinity. We study several of the connections between the properties of a multiplier in the space $\mathcal{S}'(\mathbb{R})$ and in other related spaces and the separation properties of the sequence of its zeros.

We also prove that a distributional division problem

$$Fh = f,$$

always has tempered solutions h for any tempered data f if and only if the non integrable function $1/F$ admits regularizations that are tempered, and that this holds if and only if the pseudofunction $\mathcal{P}f\,(1/F)$ is tempered.

Key Words: Tempered distributions, division problems, separation of sequences.

AMS Subject Classifications: 46F10

1 Introduction

Starting with the famous articles of Łojasiewics [15] and Hörmander [11], the study of division problems in the space of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$ and in its dual $\mathcal{S}(\mathbb{R}^n)$, has attracted the attention of several workers in the area of Functional Analysis. Indeed, the topic has been important not only in the development of the theory of distributions but also in its applications, particularly in differential equations.

In [3] Bonet, Frerick, and Jordá gave a characterization of those multiplication operators in $\mathcal{S}(\mathbb{R})$ with closed range, namely those operators for which the division problem in the space of distributions $\mathcal{S}'(\mathbb{R})$ can be solved. They showed that the zeros of such a

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multiplier have to form a sequence tending to $\pm\infty$ and that this sequence has to satisfy a special separation condition, precisely what we call (N,ν)-separation in [1] and in the present study. Their characterization also involves the rate of decrease of the multipliers.

In [8] we continued the study of such division problems, establishing –with tools from the theory of tauberian theorems for generalized functions [16, 17, 21] and the notion of Cesàro behavior of distributions [5, 7]–that for a multiplier $F \in \mathcal{O}_M(\mathbb{R})$ that never vanishes the equation

$$Fh = f, \tag{1.1}$$

has solutions $h \in \mathcal{S}'(\mathbb{R})$ for any $f \in \mathcal{S}'(\mathbb{R})$ if and only if $1/F \in \mathcal{S}'(\mathbb{R})$. In this article we are able to extend this to multipliers with zeros, showing that (1.1) has tempered solutions for any tempered f if and only if the non locally integrable function $1/F$ admits tempered distributional regularizations. In fact, we are able to show that one can always consider one specific regularization, namely, the canonical pseudofunction regularization† $\mathcal{P}f(1/F)$. The main aim of the present work is to further clarify the relationship between the separation properties of the sequence $\{x_n\}_{n=1}^{\infty}$ of zeros of F and the behavior of the multiplier not only in $\mathcal{S}(\mathbb{R})$ but in various other useful spaces of smooth functions and their duals. Following [1], we introduce, in Section 2, the notions of ν-separation and (N,ν)-separation of a sequence. In Section 3 we show that there are multipliers with closed range of the form $\sin\alpha(x)$ for some smooth increasing function α if and only if $\{x_n\}_{n=1}^{\infty}$ is ν-separated for some ν. We also show that in the case of ν-separation any multiplier F with precisely those zeros can be factored as

$$F(x) = (\sin\alpha(x))\,G(x), \tag{1.2}$$

where G is a multiplier that never vanishes. Furthermore, we show that when the sequence is not ν-separated we can find smooth functions φ with zeros at the $\{x_n\}$ and with $\varphi(x), \varphi'(x) = o\left(x^{-\lambda}\right)$ as $x \to \infty$ for each $\lambda > 0$, but such that the smooth function φ/F is not of slow growth.

In Section 5 we study the case of (N,ν)-separation, establishing and then employing a corresponding factorization,

$$F(x) = (\sin\alpha_1(x)\cdots\sin\alpha_N(x))\,G(x). \tag{1.3}$$

Among other things we prove that if the sequence is (N,ν)-separated then we can find multipliers F that give isomorphisms of $\mathcal{S}(\mathbb{R})$ onto its image, but that this is never possible if the sequence is not (N,ν)-separated for any N and ν. In fact, if $\{x_n\}$ is not (N,ν)-separated for any ν we can construct smooth functions with those zeros and with $\varphi^{(j)}(x) = o\left(x^{-\lambda}\right)$ as $x \to \infty$ for $0 \le j \le N$ for each $\lambda > 0$, but with φ/F not of slow growth.

†Pseudofunctions were introduced by Schwartz [18, Chapter 2, Section 2] at the early stages of the development of the theory of distributions.