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Liouville Type Theorems for a Class of General Parabolic Hessian Quotient Type Equations

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Abstract. We first consider the a priori estimates to a class of general parabolic (k,l)-Hessian quotient type equations of the form

$$-u_t\frac{\sigma_k(\tau\Delta uI \pm D^2u)}{\sigma_l(\tau\Delta uI \pm D^2u)} = \psi(x,t,u,Du) \quad \text{in} \quad \mathbb{R}^n\times(-\infty,0]$$

with $0\le l<k\le n$. We derive that any k-admissible-monotone solution to

$$-u_t\frac{\sigma_k(\tau\Delta uI - D^2u)}{\sigma_l(\tau\Delta uI - D^2u)} = \psi(x,t,u,Du) \quad \text{with} \quad \tau\ge 1$$

or

$$-u_t\frac{\sigma_k(\tau\Delta uI + D^2u)}{\sigma_l(\tau\Delta uI + D^2u)} = \psi(x,t,u,Du) \quad \text{with} \quad \tau> 0$$

has interior gradient estimates and Pogorelov type estimates. As an application, we prove Liouville type theorems for these equations.

Key Words: Parabolic (k,l)-Hessian quotient type equations, Pogorelov type estimates, Liouville type theorems.

AMS Subject Classifications: 35K55, 35B45, 35B08

1 Introduction

In this paper, we study a class of general parabolic (k,l)-Hessian quotient type equations

$$-u_t\frac{\sigma_k(\tau\Delta uI \pm D^2u)}{\sigma_l(\tau\Delta uI \pm D^2u)} = \psi(x,t,u,Du) \tag{1.1}$$

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in a bounded domain $D \subset \mathbb{R}^n \times (-\infty, 0]$. Here Δ is a Laplace operator, I is the $n \times n$ unit matrix, Du is the gradient of $u(x,t)$ with respect to x and $D^2u = \{u_{ij}\}$ is the matrix of second derivatives of u with respect to x. Let $u_t(x,t)$ be the derivative of u with respect to t and ψ be a smooth positive function. We define $D(t) = \{x \in \mathbb{R}^n : (x,t) \in D,\ t \leq 0\}$ and $t_0 = \inf\{t \leq 0 : D(t) \neq \emptyset\}$. The parabolic boundary $\partial_p D$ is defined by

$$\partial_p D = (\overline{D(t_0)} \times \{t_0\}) \cup \bigcup_{t \leq 0} (\partial D(t) \times \{t\}),$$

where $\overline{D(t_0)}$ is the closure of $D(t_0)$ and $\partial D(t)$ is the boundary of $D(t)$. We denote points in $\overline{D} \times \mathbb{R} \times \mathbb{R}^n$ by (x,t,z,p), where $(x,t) \in \overline{D}$, $z \in \mathbb{R}$ and $p \in \mathbb{R}^n$. τ will be determined later.

We recall the following definitions.

Definition 1.1. *Let $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n$, we set the k-th elementary symmetric polynomial*

$$\sigma_k(\lambda) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}$$

for any $k = 1, 2, \cdots, n$. We also set $\sigma_0 = 1$ and $\sigma_k = 0$ for $k > n$ or $k < 0$.

Definition 1.2. *For an open set Ω in $\mathbb{R}^n$, a function $u \in C^2(\Omega)$ is called k-admissible if $\lambda(\tau \Delta u I \pm D^2 u) \in \Gamma_k$ for any $x \in \Omega$, where Γ_k is the Gårding's cone*

$$\Gamma_k = \{\lambda \in \mathbb{R}^n : \sigma_i(\lambda) > 0,\ \forall 1 \leq i \leq k\}.$$

And, we define an open convex cone by

$$-\Gamma_k = \{\lambda \in \mathbb{R}^n : \sigma_i(\lambda) < 0,\ \forall 1 \leq i \leq k\}.$$

Moreover, we say that a function $u(x,t)$ is k-admissible-monotone, if u is k-admissible in x and decreasing in t.

Harvey-Lawson [18,19] introduced $(n-1)$-plurisubharmonic functions $u \in C^2(\mathbb{C}^n)$ satisfying the complex Hessian matrix

$$\left[\Big(\sum_{m=1}^n \frac{\partial^2 u}{\partial z_m \partial \bar{z}_m} \Big) \delta_{ij} - \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j} \right]_{1 \leq i,j \leq n} \tag{1.2}$$

is nonnegative definite. For $(n-1)$-plurisubharmonic functions, one can consider the following complex Monge-Ampère type equations

$$\det \left(\Big(\sum_{m=1}^n \frac{\partial^2 u}{\partial z_m \partial \bar{z}_m} \Big) \delta_{ij} - \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j} \right) = \psi, \tag{1.3}$$

which is related to Gauduchon conjecture in complex geometry [13,32].