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Variable Exponent Central Morrey Estimates for Multilinear Bochner-Riesz Operators

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Abstract. We prove the boundedness of multilinear Bochner-Riesz operator on central Morrey spaces with variable exponents. Moreover, we also show that the variable exponent Lipschitz commutator of Bochner-Riesz operator are bounded on above spaces. As applications, we also obtain the bounds of the commutators generated by λ-central BMO functions and Bochner-Riesz operators on central Morrey spaces with variable exponents.

Key Words: Multilinear Bochner-Riesz operator, commutator, central Morrey space with variable exponent, boundedness.

AMS Subject Classifications: 42B15, 42B20, 42B35

1 Introduction

Lebesgue spaces with variable exponent $L^{p(\cdot)}(\mathbb{R}^n)$ become one of the important class function spaces due to the seminal paper [22] by Kováčik and Rákosník. Let $p(\cdot)$ be a measurable function on $\mathbb{R}^n$ taking values in $[1,\infty)$, the Lebesgue space with variable exponent $L^{p(\cdot)}(\mathbb{R}^n)$ is defined by

$$L^{p(\cdot)}(\mathbb{R}^n) := \left\{ f \text{ is measurable on } \mathbb{R}^n : \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} \mathrm{d}x < \infty \text{ for some } \lambda > 0 \right\},$$

then $L^{p(\cdot)}(\mathbb{R}^n)$ is a Banach function space equipped with the norm

$$\|f\|_{L^{p(\cdot)}} := \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} \mathrm{d}x \le 1 \right\}.$$

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The space $L^{p(\cdot)}_{\rm loc}(\mathbb{R}^n)$ is defined by

$$L^{p(\cdot)}_{\rm loc}(\mathbb{R}^n) := \left\{ f : f\chi_K \in L^{p(\cdot)}(\mathbb{R}^n) \text{ for all compact subsets } K \subset \mathbb{R}^n \right\},$$

where and what follows, χ_S denotes the characteristic function of a measurable set $S \subset \mathbb{R}^n$. Let $p(\cdot) : \mathbb{R}^n \to (0,\infty)$, Denote by $\mathscr{P}(\mathbb{R}^n)$ the set of all measurable functions $p(\cdot) : \mathbb{R}^n \to (1,\infty)$ such that

$$1 < p^- := \operatorname{ess}\inf_{x\in\mathbb{R}^n} p(x), \quad p^+ := \operatorname{ess}\sup_{x\in\mathbb{R}^n} p(x) < \infty,$$

and $\mathscr{P}_0(\mathbb{R}^n)$ consists of all $p(\cdot)$ satisfying $p^- > 0$ and $p^+ < \infty$.

Let $f \in L^1_{\rm loc}(\mathbb{R}^n)$. Then the Hardy-Littlewood maximal operator is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)| \mathrm{d}y,$$

where

$$B_r(x) = \{y \in \mathbb{R}^n : |x-y| < r\}.$$

The set $\mathscr{B}(\mathbb{R}^n)$ consists of $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ satisfying the condition that M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. It is important to suppose that $p(\cdot)$ satisfies LH condition in the study of variable exponent function spaces.

Definition 1.1 ([1]). *Let $p(\cdot)$ be a real-valued function on $\mathbb{R}^n$, we say that $p(\cdot) \in LH$, if $p(\cdot)$ satisfies*

$$|p(x) - p(y)| \le \frac{C}{-\log|x-y|}, \quad |x-y| \le \frac{1}{2},$$

and

$$|p(x) - p(y)| \le \frac{C}{\log(|x|+e)}, \quad |y| \ge |x|.$$

It is well know that $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$ if $p(\cdot) \in \mathscr{P}(\mathbb{R}^n) \cap LH$.

The study of the boundedness properties of operators, particularly singular integral operators, holds a paramount position in the application of central bounded mean oscillation (BMO) spaces, Morrey-type spaces, and other related functional spaces [2–7]. In the academic landscape, Mizuta et al. [8] pioneered the concept of variable exponent non-homogeneous central Morrey spaces in 2015. Subsequently, in 2019, Fu et al. [9] further advanced this field by introducing central BMO spaces and central Morrey spaces with variable exponents, successfully demonstrating the boundedness of certain classical operators within these spaces. These studies provide crucial theoretical support and practical guidance for the development of operator theory.