Analysis in Theory and Applications
Anal. Theory Appl., Vol. **41**, No. 3 (2025), pp. 284-298

DOI: 10.4208/ata.OA-2024-0020
September 2025

Weighted Boundedness of Toeplitz Type Operator Associated to General Singular Integral Operator Satisfying a Variant of Hörmander's Condition

Lanzhe Liu[1,*] and Xiaoqi Zhou[1]

[1] *Department of Mathematics, School of General Education, Hunan University of Information and Technology, Changsha, Hunan 410151, China*

Received 15 July 2024; Accepted (in revised version) 25 May 2025

Abstract. In this paper, we establish the weighted sharp maximal function inequalities for the Toeplitz type operator associated to the general singular integral operator satisfying a variant of Hörmander's condition. As an application, we obtain the boundedness of the operator on weighted Lebesgue and Morrey spaces.

Key Words: Toeplitz type operator, singular integral operator, sharp maximal function, weighted *BMO*, weighted Lipschitz function.

AMS Subject Classifications: 42B20, 42B25

1 Introduction and preliminaries

As the development of singular integral operators (see [8, 23]), their commutators and multilinear operators have been well studied. In [3, 21, 22], the authors prove that the commutators generated by the singular integral operators and *BMO* functions are bounded on $L^p(R^n)$ for $1 < p < \infty$. Chanillo (see [2]) proves a similar result when singular integral operators are replaced by the fractional integral operators. In [11, 18], the boundedness for the commutators generated by the singular integral operators and Lipschitz functions on Triebel-Lizorkin and $L^p(R^n)$ $(1 < p < \infty)$ spaces are obtained. In [1, 10], the boundedness for the commutators generated by the singular integral operators and the weighted *BMO* and Lipschitz functions on $L^p(R^n)$ $(1 < p < \infty)$ spaces are obtained. In [13, 15, 16], some Toeplitz type operators related to the singular integral operators and strongly singular integral operators are introduced, and the boundedness for the operators generated by *BMO* and Lipschitz functions are obtained. In [9], some singular integral operators satisfying a variant of Hörmander's condition are introduced, and the boundedness for the operators are obtained (see [9, 24, 25]). Motivated by these,

*Corresponding author. *Email addresses:* lanzheliu@163.com (L. Liu), 53245219@qq.com (X. Zhou)

in this paper, we will study the Toeplitz type operator associated to the general singular integral operators satisfying a variant of Hörmander's condition and the weighted Lipschitz and *BMO* functions.

First, let us introduce some notations. Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For any locally integrable function f, the sharp maximal function of f is defined by

$$M^{\#}(f)(x)=\sup_{Q\ni x}\frac{1}{|Q|}\int_Q|f(y)-f_Q|dy,$$

where, and in what follows,

$$f_Q=|Q|^{-1}\int_Q f(x)dx.$$

It is well-known that (see [8,23])

$$M^{\#}(f)(x)\approx\sup_{Q\ni x}\inf_{c\in C}\frac{1}{|Q|}\int_Q|f(y)-c|dy.$$

Let

$$M(f)(x)=\sup_{Q\ni x}\frac{1}{|Q|}\int_Q|f(y)|dy.$$

For $\eta>0$, let

$$M_\eta^{\#}(f)(x)=M^{\#}(|f|^\eta)^{1/\eta}(x)\quad\text{and}\quad M_\eta(f)(x)=M(|f|^\eta)^{1/\eta}(x).$$

For $0<\eta<n$, $1\le p<\infty$ and the non-negative weight function w, set

$$M_{\eta,p,w}(f)(x)=\sup_{Q\ni x}\left(\frac{1}{w(Q)^{1-p\eta/n}}\int_Q|f(y)|^p w(y)dy\right)^{1/p}.$$

We write $M_{\eta,p,w}(f)=M_{p,w}(f)$ if $\eta=0$. The A_p weight is defined by (see [8]), for $1<p<\infty$,

$$A_p=\left\{0<w\in L^1_{loc}(R^n):\sup_Q\left(\frac{1}{|Q|}\int_Q w(x)dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}dx\right)^{p-1}<\infty\right\}$$

and

$$A_1=\{0<w\in L^p_{loc}(R^n):M(w)(x)\le Cw(x),a.e.\}.$$

Given a non-negative weight function w. For $1\le p<\infty$, the weighted Lebesgue space $L^p(R^n,w)$ is the space of functions f such that

$$||f||_{L^p(w)}=\left(\int_{R^n}|f(x)|^p w(x)dx\right)^{1/p}<\infty.$$