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Intrinsic Square Functions and Commutators on Generalized Morrey Spaces Associated with Ball Banach Function Spaces

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Abstract. In this paper, the authors study the boundedness for a large class of intrinsic square functions and their commutators on generalized Morrey spaces associated with ball Banach function spaces, respectively.

Key Words: Intrinsic square function, commutator, generalized Morrey space, ball Banach function space.

AMS Subject Classifications: 42B20, 42B25

1 Introduction

It is well known that the theory of Littlewood-Paley square functions plays an important role in harmonic analysis, such as in the study of Fourier multiplier and singular integral operators. In 2007, Wilson [16] first introduced some new square function called intrinsic square function which is universal in a sense (see also [17]). In [16], Wilson proved the intrinsic square functions is bounded from $L^p(\mathbb{R}^n)$ to itself and also extended the weighted case. In 2012, Wang [12] obtained the boundedness of intrinsic square functions on weighted Morrey spaces, and their commutators generated with BMO functions on weighted Lebesgue and Morrey spaces are established, respectively. Later on, Guliyev and Shukurov generalized the results in [6] to the generalized Morrey spaces. For many rich achievements and further developments in this subject, we refer the readers to [9,12,13,16] and references therein.

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The main purpose in this paper is to establish the boundedenss of intrinsic square functions and their commutators on generalized Morrey spaces associated with ball Banach function spaces, respectively. The generalized Morrey spaces associated with ball Banach function spaces were introduced in [18], which include a number of Morrey type spaces such as the generalized weighted Morrey spaces, the generalized Morrey with variable exponents spaces, the mixed-norm Morrey spaces, the generalized Orlicz-Morrey spaces as special cases. Therefore, our results presented in this paper can be regarded as unity and extension of many known results related to intrinsic square functions.

The rest of this paper is organized as follows. In Section 2, we recall some definitions and preliminaries related to the current work. The boundedenss of intrinsic square functions and their commutators on generalized Morrey spaces associated with ball Banach function spaces will be obtained in Sections 3 and 4, respectively. Finally, we will give some applications of main results in Section 5.

2 Some preliminaries

For $0 < \alpha \le 1$, let $\mathcal{C}_\alpha$ be the family of functions $\phi : \mathbb{R}^n \to \mathbb{R}$ such that ϕ has support containing in $\{x \in \mathbb{R}^n : |x| \le 1\}$, $\int_{\mathbb{R}^n} \phi(x)dx = 0$, and for all $x, x' \in \mathbb{R}^n$,

$$|\phi(x) - \phi(x')| \le |x - x'|^\alpha.$$

For $(y,t) \in \mathbb{R}^{n+1}_+$ and $f \in L^1_{loc}(\mathbb{R}^n)$, set

$$\mathcal{A}_\alpha f(t,y) = \sup_{\phi \in \mathcal{C}_\alpha} |f * \phi_t(y)|,$$

where $\phi_t(y) = \frac{1}{t^n}\phi(\frac{y}{t})$.

The varying-aperture intrinsic square (intrinsic Lusin) function is defined by

$$\mathcal{G}_{\alpha,\beta}(f)(x) = \Big(\iint_{\Gamma_{\beta(x)}} \big(\mathcal{A}_\alpha f(t,y)\big)^2 \frac{dydt}{t^{n+1}} \Big)^{1/2},$$

where $\Gamma_{\beta(x)}$ denotes the cone of aperture β for any $\beta > 0$:

$$\Gamma_{\beta(x)} = \{(y,t) \in \mathbb{R}^{n+1}_+ : |x - y| < \beta t\}.$$

If $\beta \equiv 1$, we denote $\mathcal{G}_{\alpha,1}(f)$ by $\mathcal{G}_\alpha(f)$. Moreover, for any $0 < \alpha \le 1$ and $\beta \ge 1$, there is a pointwise relation between the function $\mathcal{G}_{\alpha,\beta}(f)(x)$ and $\mathcal{G}_\alpha(f)(x)$ as follows:

$$\mathcal{G}_{\alpha,\beta}(f)(x) \le \beta^{\frac{3n}{2}+\alpha}\mathcal{G}_\alpha(f)(x),$$

please refer to [16] for details.