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Blowing Up Solutions to Slightly Sub- or Super-Critical Lane-Emden Systems with Neumann Boundary Conditions

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Abstract. We prove that, for some suitable smooth bounded domain, there exists a solution to the following Neumann problem for the Lane-Emden system:

$$\begin{cases} -\Delta u_1 + \mu u_1 = u_2^{p+\alpha\varepsilon} & \text{in } \Omega, \\ -\Delta u_2 + \mu u_2 = u_1^{q+\beta\varepsilon} & \text{in } \Omega, \\ \dfrac{\partial u_1}{\partial n} = \dfrac{\partial u_2}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is some smooth bounded domain in $\mathbb{R}^N$, $N \geq 4$, $\mu > 0$, $\alpha > 0$, $\beta > 0$ are constants and $\varepsilon \neq 0$ is a small number. We show that there exists a solution to the slightly supercritical problem for $\varepsilon > 0$, and for $\varepsilon < 0$, there also exists a solution to the slightly subcritical problem if the domain is not convex.

Comparing with the single elliptic equations, the challenges and novelty are manifested in the construction of good approximate solutions characterizing the boundary behavior under Neumann boundary conditions, in which process, the selection of the range of nonlinear coupling exponents and the weighted Sobolev spaces requires elaborate discussion.

Key Words: Lane-Emden system, Neumann problem, blow up solutions, reduction method.

AMS Subject Classifications: 35J57, 35B25, 35B38

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1 Introduction and main results

In this paper, we are concerned with the following elliptic system

$$\begin{cases} -\Delta u_1 + \mu u_1 = u_2^{p+\alpha\varepsilon} & \text{in } \Omega, \\ -\Delta u_2 + \mu u_2 = u_1^{q+\beta\varepsilon} & \text{in } \Omega, \\ \dfrac{\partial u_1}{\partial n} = \dfrac{\partial u_2}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ with $N \geq 4$, μ, α, β are positive constants and $\varepsilon \neq 0$ is a small number, $p, q \in (1, \infty)$ satisfy

$$\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N}. \tag{1.2}$$

Without loss of generality, we always assume that $p \leq \frac{N+1}{N-1} \leq q$.

The system (1.1) is essentially a perturbation of the classical Lane-Emden system. In fact, under the following rescaling

$$(u_1, u_2)(x) \quad \to \quad \left(\varepsilon^{-\frac{2(1+p+\alpha\varepsilon)}{(p+\alpha\varepsilon)(q+\beta\varepsilon)-1}} u_1, \varepsilon^{-\frac{2(1+q+\beta\varepsilon)}{(p+\alpha\varepsilon)(q+\beta\varepsilon)-1}} u_2 \right) \left(\frac{x}{\varepsilon} \right), \tag{1.3}$$

the system (1.1) is equivalent to the following system

$$\begin{cases} -\Delta u_1 + \mu\varepsilon^2 u_1 = u_2^{p+\alpha\varepsilon} & \text{in } \Omega_\varepsilon, \\ -\Delta u_2 + \mu\varepsilon^2 u_2 = u_1^{q+\beta\varepsilon} & \text{in } \Omega_\varepsilon, \\ \dfrac{\partial u_1}{\partial n} = \dfrac{\partial u_2}{\partial n} = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{1.4}$$

where $\Omega_\varepsilon = \{x : \varepsilon x \in \Omega\}$. The limit system is the classical Lane-Emden system with critical exponents:

$$\begin{cases} -\Delta u_1 = |u_2|^{p-1} u_2 & \text{in } \mathbb{R}^N, \\ -\Delta u_2 = |u_1|^{q-1} u_1 & \text{in } \mathbb{R}^N. \end{cases} \tag{1.5}$$

Thanks to [17] and [33], the positive ground state $(U_{0,1}, V_{0,1})$ of (1.5) is unique with $U_{0,1}(0) = 1$ and for any $\lambda > 0$, $a \in \mathbb{R}^N$, the family of functions

$$(U_{a,\lambda}(y), V_{a,\lambda}(y)) = \left(\lambda^{\frac{N}{q+1}} U_{0,1}(\lambda(y-a)), \lambda^{\frac{N}{p+1}} V_{0,1}(\lambda(y-a)) \right),$$

also solves system (1.5). The properties of vector solutions of (1.5) is well known. For completeness, we will introduce them in Section 2 of this present work.

For decades, there are lots of results on Dirichlet problems for Schördinger equations and systems. Wherein, the Hamiltonian type of systems is highly concerned, refer to [4, 8, 10–13, 18, 19], etc. Most of them obtain solutions concentrating at a point which is