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Blowing Up Solutions to Slightly Sub- or Super-Critical Lane-Emden Systems with Neumann Boundary Conditions

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Abstract. We prove that, for some suitable smooth bounded domain, there exists a solution to the following Neumann problem for the Lane-Emden system:

$$\begin{cases} -\Delta u_1 + \mu u_1 = u_2^{p+\alpha\varepsilon} & \text{in } \Omega, \\ -\Delta u_2 + \mu u_2 = u_1^{q+\beta\varepsilon} & \text{in } \Omega, \\ \dfrac{\partial u_1}{\partial n} = \dfrac{\partial u_2}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is some smooth bounded domain in $\mathbb{R}^N$, $N \geq 4$, $\mu > 0$, $\alpha > 0$, $\beta > 0$ are constants and $\varepsilon \neq 0$ is a small number. We show that there exists a solution to the slightly supercritical problem for $\varepsilon > 0$, and for $\varepsilon < 0$, there also exists a solution to the slightly subcritical problem if the domain is not convex.

Comparing with the single elliptic equations, the challenges and novelty are manifested in the construction of good approximate solutions characterizing the boundary behavior under Neumann boundary conditions, in which process, the selection of the range of nonlinear coupling exponents and the weighted Sobolev spaces requires elaborate discussion.

Key Words: Lane-Emden system, Neumann problem, blow up solutions, reduction method.

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1 Introduction and main results

In this paper, we are concerned with the following elliptic system

$$\begin{cases} -\Delta u_1 + \mu u_1 = u_2^{p+\alpha\varepsilon} & \text{in } \Omega, \\ -\Delta u_2 + \mu u_2 = u_1^{q+\beta\varepsilon} & \text{in } \Omega, \\ \dfrac{\partial u_1}{\partial n} = \dfrac{\partial u_2}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ with $N \geq 4$, μ, α, β are positive constants and $\varepsilon \neq 0$ is a small number, $p, q \in (1, \infty)$ satisfy

$$\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N}. \tag{1.2}$$

Without loss of generality, we always assume that $p \leq \frac{N+1}{N-1} \leq q$.

The system (1.1) is essentially a perturbation of the classical Lane-Emden system. In fact, under the following rescaling

$$(u_1, u_2)(x) \quad \rightarrow \quad \left(\varepsilon^{-\frac{2(1+p+\alpha\varepsilon)}{(p+\alpha\varepsilon)(q+\beta\varepsilon)-1}} u_1, \varepsilon^{-\frac{2(1+q+\beta\varepsilon)}{(p+\alpha\varepsilon)(q+\beta\varepsilon)-1}} u_2\right)\left(\frac{x}{\varepsilon}\right), \tag{1.3}$$

the system (1.1) is equivalent to the following system

$$\begin{cases} -\Delta u_1 + \mu\varepsilon^2 u_1 = u_2^{p+\alpha\varepsilon} & \text{in } \Omega_\varepsilon, \\ -\Delta u_2 + \mu\varepsilon^2 u_2 = u_1^{q+\beta\varepsilon} & \text{in } \Omega_\varepsilon, \\ \dfrac{\partial u_1}{\partial n} = \dfrac{\partial u_2}{\partial n} = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{1.4}$$

where $\Omega_\varepsilon = \{x : \varepsilon x \in \Omega\}$. The limit system is the classical Lane-Emden system with critical exponents:

$$\begin{cases} -\Delta u_1 = |u_2|^{p-1} u_2 & \text{in } \mathbb{R}^N, \\ -\Delta u_2 = |u_1|^{q-1} u_1 & \text{in } \mathbb{R}^N. \end{cases} \tag{1.5}$$

Thanks to [17] and [33], the positive ground state $(U_{0,1}, V_{0,1})$ of (1.5) is unique with $U_{0,1}(0) = 1$ and for any $\lambda > 0$, $a \in \mathbb{R}^N$, the family of functions

$$(U_{a,\lambda}(y), V_{a,\lambda}(y)) = \left(\lambda^{\frac{N}{q+1}} U_{0,1}(\lambda(y-a)), \lambda^{\frac{N}{p+1}} V_{0,1}(\lambda(y-a))\right),$$

also solves system (1.5). The properties of vector solutions of (1.5) is well known. For completeness, we will introduce them in Section 2 of this present work.

For decades, there are lots of results on Dirichlet problems for Schördinger equations and systems. Wherein, the Hamiltonian type of systems is highly concerned, refer to [4, 8, 10–13, 18, 19], etc. Most of them obtain solutions concentrating at a point which is determined by the corresponding Green's function, but the location of the concentration points for Neumann boundary value problem is much more complex. For the following Neumann problem

$$\begin{cases} \varepsilon\Delta u - u + u^p = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

with p is a subcritical exponent, Ni and Takagi [25] proved that u_ε has only one local maximum, hence the global maximum over $\bar{\Omega}$ and it is achieved at exactly one point P_ε which must lie on the boundary, provided that ε is sufficiently small. Moreover, $u_\varepsilon \to 0$ as $\varepsilon \to 0$. Subsequently, in [26], they further locate the maximum P_ε and give an accurate description of u_ε. In particular, they proved $\max u_\varepsilon$ is achieved at P_ε satisfying $\lim_{\varepsilon\to 0} H(P_\varepsilon) = \max_{P\in\partial\Omega} H(P)$, where $H(P)$ denotes the mean curvature of $\partial\Omega$ at P. This known result is extended by Cao and Kupper in [5], Cao and Kupper studied the existence of multipeaked solutions with several local maximum points on the boundary and condensing in a small neighborhood of those points as ε approaches 0. They showed the effect of strict local maximum or minimum points of the mean curvature $H(P)$ on $\partial\Omega$ on the existence of multipeaked solutions. For p is a critical exponent, we refer to [1,32]. For other high energy solutions, we also can refer to [7,14,15,21,37] for subcritical case and [2,16,23,28,34–36] for critical case. In [30], Rey and Wei considered the following almost critical Neumann problem:

$$\begin{cases} -\Delta u + \mu u = u^{\frac{N+2}{N-2}+\varepsilon} & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

where $\varepsilon \neq 0$ is sufficiently small, $N \geq 4$. They showed that the effect of sign of perturbing parameter ε on location of concentrate points. In precise, for $\varepsilon > 0$, there always exists a solution to the slightly supercritical problem, which blows up at the most curved part of the boundary as ε goes to zero. On the other hand, for $\varepsilon < 0$, assuming that the domain is not convex, there also exists a solution to the slightly subcritical problem, which blows up at the least curved part of the domain. Note that the second case agrees with the necessity result of Gui and Lin in [16].

It is natural to inquire whether there exists a vector solution concentrating at the boundary for the Neumann problem of a strongly coupled Hamiltonian system. Currently, there are few results on this topic, which forms the primary focus of our study in this paper.

In contrast to the Dirichlet problem, although it is believed that the Neumann problem allows the existence of smooth solutions on the critical hyperbola, there have been very few results to (1.1) with Neumann conditions up to now. The existence of least energy nodal solutions has been confirmed only in the subcritical [31] and critical cases [27], characterized by the condition $\frac{n}{p+1} + \frac{n}{q+1} \geq n-2$. A significant departure from Dirichlet problems lies in the application of the Lyapunov-Schmidt reduction strategy. Specifically, solutions to Neumann problems may exhibit extrema at distinct boundary points, contrasting with Dirichlet problems where such extrema occur within the domain's interior [19, 20]. This distinction necessitates methodological adaptations, particularly due to the boundary curvature's influence and the blow-up analysis, which culminates in a scaling-limited problem within a half-space.

Motivated by [30], we conjecture the existence of vector solutions (u_1, u_2) for (1.1), where each component concentrates on the boundary, especially when $p > \frac{N}{N-2}$. This belief stems from the decay behavior $r^{-(N-2)}$ of each component in (1.5) under such conditions. However, the blow-up dynamics differ significantly from those of the single Lane-Emden equation, posing challenges due to the non-explicit nature of $(U_{0,1}(y), V_{0,1}(y))$ and the limited information available beyond asymptotic decay. These complexities underscore the need for innovative approaches and deeper analysis to overcome these inherent difficulties.

Let us outline the proof idea and introduce the main theorems in this paper.

We aim to construct a bubble solution concentrating on the boundary by using the Lyaponouv reduction method. In the first step, we need good approximation solutions. Then we choose some suitable function space to ensure the invertibility of the linearized operator for the approximate solutions in the space. The last step is to discuss the reduced problem and find the critical point of the corresponding energy functional defined in $(W^{2,\frac{p+1}{p}} \cap W^{1,p^*}) \times (W^{2,\frac{q+1}{q}} \cap W^{1,q^*})$:

$$J_\varepsilon(u_1, u_2) := \int_{\Omega_\varepsilon} \nabla u_1 \cdot \nabla u_2 + \mu\varepsilon^2 \int_{\Omega_\varepsilon} u_1 u_2 - \frac{1}{p+\alpha\varepsilon+1} \int_{\Omega_\varepsilon} |u_2|^{p+\alpha\varepsilon+1} - \frac{1}{q+\beta\varepsilon+1} \int_{\Omega_\varepsilon} |u_1|^{q+\beta\varepsilon+1}.$$

We first explain how to find a good approximation vector solution and to ensure the nontrivial critical points of $J(u_1, u_2)$. Noticing that the limit problem of (1.1) (or (1.4)) is (1.5), it is natural to consider the projection (W_1, W_2) as the approximate vector solution, which solves

$$\begin{cases} -\Delta W_1 + \mu\varepsilon^2 W_1 = V^p_{\xi,\lambda} & \text{in } \Omega_\varepsilon, \\ -\Delta W_2 + \mu\varepsilon^2 W_2 = U^q_{\xi,\lambda} & \text{in } \Omega_\varepsilon, \\ \dfrac{\partial W_1}{\partial n} = \dfrac{\partial W_2}{\partial n} = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases} \tag{1.8}$$

Then, the crucial step is the expansion of W_1 and W_2. In order to estimate the difference of $W_1 - U_{\xi,\lambda}$ and $W_2 - V_{\xi,\lambda}$, considering the Neumann boundary conditions, we introduce the auxiliary functions $\varphi_{1,0}$ and $\varphi_{2,0}$. These functions are related not only to the principal curvatures at ξ and thus the mean curvature $H(\xi)$, but also to the boundary behavior of the ground state solution. For further details, see Appendix A.

The next crucial step is to calculate the energy of the approximate vector solution. In Appendix B, integral estimates suggest making a priori assumptions that λ behaves as $\mathcal{O}(\varepsilon^0)$ as ε tends to zero. Moreover, these integral estimates also indicate that the locations of the critical points of $J(W_1, W_2)$ are highly correlated with the mean curvature on the boundary.

For our workspaces, motivated by the Corollary 1 of [24], we define some weighted Sobolev spaces endowed with norm of

$$||\phi||_{W^{l,t}_{\gamma}(D)} = \sum_{|d|=0}^{l} ||\langle x-\xi\rangle^{\gamma+|d|}\partial^d\phi||_{L^t(D)}, \tag{1.9}$$

where D is an open domain in $\mathbb{R}^N$, l is a nonnegative integer, $t>1$, γ are real numbers, $\xi \in D$ and $\langle x-\xi\rangle = (1+|x-\xi|^2)^{\frac{1}{2}}$. The integral operator

$$Tu(x) = \int_{\mathbb{R}^N} \frac{u(y)}{|x-y|^{N-2}} dy \tag{1.10}$$

is a bounded operator from $L^t_{\gamma+2}(\mathbb{R}^N)$ to $L^t_{\gamma}(\mathbb{R}^N)$, provided that $-\frac{N}{t} < \gamma < \frac{N}{t'} - 2$, where $\frac{N}{t} + \frac{N}{t'} = 1$, $L^t_{\gamma}(\mathbb{R}^N)$ is defined as (1.9). For any function f in Ω_ε, denote

$$||f||_{*,i} = ||f||_{W^{2,t_i}_{\gamma_i}(\Omega_\varepsilon)}, \tag{1.11a}$$

$$||f||_{**,i} = ||f||_{L^{t_i}_{\gamma_i}(\Omega_\varepsilon)}, \tag{1.11b}$$

where t_i, γ_i, $i=1,2$ are to be determined later.

The rest of proof follows the standard reduction methods. With sophisticated calculations and precise analysis, we impose the following assumption (A) on exponent p and parameters t_i, γ_i, $i=1,2$.

Assumption (A). $t_1 > N$, $\frac{N-2}{2} + \frac{N(N-2)}{4t_1} < \gamma_1 < N-2-\frac{N}{t_1}$, and

$$N \geq 5, \quad p \in \left(\frac{N-1+\frac{N}{t_2}}{N-3}, \frac{N+2}{N-2}\right), \quad t_2 > \max\left\{N, \frac{N}{2-\frac{4}{N-2}}\right\}, \quad \frac{2+\frac{N}{t_2}}{p-1} < \gamma_2 < N-3.$$

The main theorems of this work are as follows.

Theorem 1.1. *Under the assumptions* (1.2) *and* (A), *the system* (1.1) *has a nontrivial solution for $\varepsilon > 0$ close enough to zero, which blows up as ε goes to zero at a point $a \in \partial\Omega$, such that $H(a) = \max_{P\in\partial\Omega} H(P)$.*

Theorem 1.2. *Under the assumptions* (1.2) *and* (A), *assume that Ω is not convex, the system* (1.1) *has a nontrivial solution for $\varepsilon < 0$ close enough to zero, which blows up as ε goes to zero at a point $a \in \partial\Omega$, such that $H(a) = \min_{P\in\partial\Omega} H(P)$.*

Let us gives some remarks on the assumption (A).

Remark 1.1. 1). To ensure the operator T, defined as (1.10), is bounded, we always assume that $\gamma_i < \frac{N}{t_i'} - 2 = N - 2 - \frac{N}{t_i}$.

2). Once we study the inversion of the linearized problem, we need impose $t_i > N$ so that the Sobolev embedding theorem is valid, and we can obtain the pointwise estimates on ϕ_i and $\nabla\phi_i$, $i = 1, 2$, see Lemma 3.1.

3). With the conditions in 1) and 2), the inversion of the linearized problem holds only for $p > \frac{N+\frac{N}{t_2}}{N-2}$, see Lemma 3.1.

4). The further determination of the range of p, γ_i, and t_i stems from the accurately calculations on R_ε and N_ε, see Lemmas 3.2 and 3.3. By the way, we give some necessary limit on the dimension of space.

Remark 1.2. We emphasize that the method involving nonlinear projection, as used in [19] and [10] to handle small $p < \frac{N}{N-2}$ is not applicable to our problem. This is because the concentration points are located on the boundary of the domain, where the regular part of the nonlinear Green's function $\tilde{G}$ becomes very large and problematic. In fact, we can even establish non-existence results in certain special cases when p is small and blow-up occurs on $\partial\Omega$. These are precisely the issues we are currently investigating. Therefore, at least in this context, assumption (A) may not be relaxed for the study of concentrated solutions to Neumann boundary value problems.

The main difficulties of the proof come from the complex relationship between the exponents p, q of the strongly coupled nonlinear terms in the system and the selection of parameters t_1, t_2, γ_1, γ_2 involving the suitable weighted Sobolev spaces, as well as the construction and expansion estimates of the projection function W_1, W_2 corresponding to Neumann boundary conditions in the process of selecting appropriate approximate solutions.

This paper is organized as follows. In Section 2, we list the properties of vector solutions of (1.5) and give a counterpart of the Theorem 1.1. We carry out the standard Lyaponov reduction in Section 3, while the reduced finite dimensional problem is solved in Section 4. In the appendix, we give the expansion of approximate solutions, the definition of the mean curvature and the energy estimates of approximate solutions.

2 Preliminaries

2.1 The ground-state solutions to Lane-Emden systems

We first introduce the limit problem (1.5). The positive ground state (U, V) to the following system was found in [22],

$$\begin{cases} -\Delta U = |V|^{p-1}V & \text{in } \mathbb{R}^N, \\ -\Delta V = |U|^{q-1}U & \text{in } \mathbb{R}^N, \\ (U, V) \in \dot{W}^{2,\frac{p+1}{p}}(\mathbb{R}^N) \times \dot{W}^{2,\frac{q+1}{q}}(\mathbb{R}^N), \end{cases} \tag{2.1}$$

where $N \geq 3$ and (p,q) satisfy

$$\frac{1}{p+1}+\frac{1}{q+1}=\frac{N-2}{N}. \tag{2.2}$$

By Sobolev embeddings, there holds that

$$\begin{cases} \dot{W}^{2,\frac{p+1}{p}}(\mathbb{R}^N) \hookrightarrow \dot{W}^{1,p^*}(\mathbb{R}^N) \hookrightarrow L^{q+1}(\mathbb{R}^N), \\ \dot{W}^{2,\frac{q+1}{q}}(\mathbb{R}^N) \hookrightarrow \dot{W}^{1,q^*}(\mathbb{R}^N) \hookrightarrow L^{p+1}(\mathbb{R}^N), \end{cases} \tag{2.3}$$

with

$$\frac{1}{p^*}=\frac{p}{p+1}-\frac{1}{N}=\frac{1}{q+1}+\frac{1}{N},\quad \frac{1}{q^*}=\frac{q}{q+1}-\frac{1}{N}=\frac{1}{p+1}+\frac{1}{N},$$

and so the following energy functional is well-defined in $\dot{W}^{2,\frac{p+1}{p}}(\mathbb{R}^N)\times\dot{W}^{2,\frac{q+1}{q}}(\mathbb{R}^N)$:

$$I_0(u,v):=\int_{\mathbb{R}^N}\nabla u\cdot\nabla v-\frac{1}{p+1}\int_{\mathbb{R}^N}|v|^{p+1}-\frac{1}{q+1}\int_{\mathbb{R}^N}|u|^{q+1}.$$

According to [3], the ground state is radially symmetric and decreasing up to a suitable translation. Thanks to [17] and [33], the positive ground state $(U_{0,1},V_{0,1})$ of (2.1) is unique with $U_{0,1}(0)=1$ and for any $\lambda>0, a\in\mathbb{R}^N$, the family of functions

$$(U_{a,\lambda}(y),V_{a,\lambda}(y))=\left(\lambda^{\frac{N}{q+1}}U_{0,1}(\lambda(y-a)),\lambda^{\frac{N}{p+1}}V_{0,1}(\lambda(y-a))\right)$$

also solves system (2.1). Sharp asymptotic behavior of the ground states to (2.1) (see [17]) and the non-degeneracy for (2.1) at each ground state (see [9]) play an important role in constructing bubbling solutions especially when using the Lyapunov-Schmidt reduction method.

More precisely, the bubbles satisfy the following properties.

Proposition 2.1 (c.f. [17])**.** *some positive constants $a=a_{N,p}$ and $b=b_{N,p}$ depending only on N and p such that*

$$\lim_{r\to\infty} r^{N-2}V_{0,1}(r)=b_{N,p}, \tag{2.4}$$

while

$$\begin{cases} \lim\limits_{r\to\infty} r^{N-2}U_{0,1}(r)=a_{N,p}, & \text{if } p>\dfrac{N}{N-2}, \\ \lim\limits_{r\to\infty} \dfrac{r^{N-2}}{\log r}U_{0,1}(r)=a_{N,p}, & \text{if } p=\dfrac{N}{N-2}, \\ \lim\limits_{r\to\infty} r^{(N-2)p-2}U_{0,1}(r)=a_{N,p}, & \text{if } p<\dfrac{N}{N-2}, \end{cases} \tag{2.5}$$

where in the last case, we have $b^p_{N,p}=a_{N,p}((N-2)p-2)(N-(N-2)p)$.

Proposition 2.2 (c.f. [9]). *Set*

$$(\Psi^0_{0,1},\Phi^0_{0,1})=\Big(y\cdot\nabla U_{0,1}+\frac{NU_{0,1}}{q+1},y\cdot\nabla V_{0,1}+\frac{NV_{0,1}}{p+1}\Big)$$

and

$$(\Psi^l_{0,1},\Phi^l_{0,1})=(\partial_{y_l}U_{0,1},\partial_{y_l}V_{0,1})\quad for\ \ l=1,\cdots,N.$$

Then the space of solutions to the linear system

$$\begin{cases}-\Delta\Psi=pV_{0,1}^{p-1}\Phi & in\ \ \mathbb{R}^N,\\ -\Delta\Phi=qU_{0,1}^{q-1}\Psi & in\ \ \mathbb{R}^N,\\ (\Psi,\Phi)\in\dot{W}^{2,\frac{p+1}{p}}(\mathbb{R}^N)\times\dot{W}^{2,\frac{q+1}{q}}(\mathbb{R}^N),\end{cases}$$

is spanned by

$$\Big\{(\Psi^0_{0,1},\Phi^0_{0,1}),(\Psi^1_{0,1},\Phi^1_{0,1}),\cdots,(\Psi^N_{0,1},\Phi^N_{0,1})\Big\}.$$

For sake of simplicity, we consider in the following the supercritical case, i.e., we assume that $\varepsilon>0$. The subcritical case may be treated exactly in the same way. As $a\in\partial\Omega$ and λ goes to infinity, the solutions of (2.1) provide us with approximate solutions to the problem that we are interested in. However, in view of the additional linear terms μu_1 and μu_2, the approximation needs to be improved.

2.2 A counterpart of the Theorem 1.1

Fix $a\in\partial\Omega$. We define projection (v_1,v_2) satisfying

$$\begin{cases}-\Delta v_1+\mu v_1=V^p_{a,\lambda} & \text{in }\ \Omega,\\ -\Delta v_2+\mu v_2=U^q_{a,\lambda} & \text{in }\ \Omega,\\ \dfrac{\partial v_1}{\partial n}=\dfrac{\partial v_2}{\partial n}=0 & \text{on }\ \partial\Omega.\end{cases}\tag{2.6}$$

In the neighborhood of the suitable approximate vector solution (v_1,v_2), we are to find a true solution to the problem (1.1).

Through the meticulous calculations of energy of (v_1,v_2) (refer to Appendix C), we have additional a priori assumption that λ behaves as $\frac{1}{\varepsilon}$ when ε goes to zero. We thus set

$$\lambda=\frac{1}{\Lambda\varepsilon},\quad \frac{1}{\delta'}<\Lambda<\delta',\tag{2.7}$$

with δ' some strictly positive number.

In order to make further computations easier, we proceed to a rescaling. We set

$$\Omega_\varepsilon=\frac{\Omega}{\varepsilon},\quad \xi=\frac{a}{\varepsilon},\quad (W_1(x),W_2(x))=(\varepsilon^{\frac{N}{q+1}}v_1(\varepsilon x),\varepsilon^{\frac{N}{p+1}}v_2(\varepsilon x)).\tag{2.8}$$

Then we have

$$\begin{cases} -\Delta W_1 + \mu\varepsilon^2 W_1 = V^p_{\xi,\frac{1}{\Lambda}} & \text{in } \Omega_\varepsilon, \\ -\Delta W_2 + \mu\varepsilon^2 W_2 = U^q_{\xi,\frac{1}{\Lambda}} & \text{in } \Omega_\varepsilon, \\ \dfrac{\partial W_1}{\partial n} = \dfrac{\partial W_2}{\partial n} = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases} \tag{2.9}$$

On the other hand, through the following rescaling

$$(u_1, u_2)(x) \quad \to \quad \left(\varepsilon^{-\frac{2(1+p+\alpha\varepsilon)}{(p+\alpha\varepsilon)(q+\beta\varepsilon)-1}} u_1, \varepsilon^{-\frac{2(1+q+\beta\varepsilon)}{(p+\alpha\varepsilon)(q+\beta\varepsilon)-1}} u_2\right)\left(\frac{x}{\varepsilon}\right), \tag{2.10}$$

the system (1.1) is equivalent to (1.4):

$$\begin{cases} -\Delta u_1 + \mu\varepsilon^2 u_1 = u_2^{p+\alpha\varepsilon} & \text{in } \Omega_\varepsilon, \\ -\Delta u_2 + \mu\varepsilon^2 u_2 = u_1^{q+\beta\varepsilon} & \text{in } \Omega_\varepsilon, \\ \dfrac{\partial u_1}{\partial n} = \dfrac{\partial u_2}{\partial n} = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases}$$

Therefore, finding a solution to (1.1) in a neighborhood of (v_1, v_2) is equivalent to finding a solution to (1.4) in a neighborhood of (W_1, W_2).

We verify Theorem 1.1 by prove the following theorem.

Theorem 2.1. *Problem* (1.4) *has a solution* $(u_1, u_2) \in (W^{2,\frac{p+1}{p}} \cap W^{1,p^*})(\Omega_\varepsilon) \times (W^{2,\frac{q+1}{q}} \cap W^{1,q^*})(\Omega_\varepsilon)$ *of the form*

$$u_1 = W_1 + \omega_1, \quad u_2 = W_2 + \omega_2, \tag{2.11}$$

with (ω_1, ω_2) *small and orthogonal at* (W_1, W_2), *in a suitable sense, to the manifold*

$$M = \{(W_1, W_2),\ \Lambda \text{ satisfying } (2.7),\ \xi \in \partial\Omega_\varepsilon\}.$$

3 Finite-dimensional reduction

In this section, we perform a finite-dimensional reduction.

In Appendix A, we derive the following asymptotic expansion of (v_1, v_2): For $p > \frac{N}{N-2}$, $N \geq 4$, we have the expansion

$$\begin{cases} v_1 = U_{a,\frac{1}{\Lambda\varepsilon}} - (\Lambda\varepsilon)^{1-\frac{N}{q+1}} \varphi_{1,0}\left(\dfrac{x-a}{\Lambda\varepsilon}\right) + \mathcal{O}(\varepsilon^{2-\frac{N}{q+1}} |\ln\varepsilon|^m), \\ v_2 = V_{a,\frac{1}{\Lambda\varepsilon}} - (\Lambda\varepsilon)^{1-\frac{N}{p+1}} \varphi_{2,0}\left(\dfrac{x-a}{\Lambda\varepsilon}\right) + \mathcal{O}(\varepsilon^{2-\frac{N}{p+1}} |\ln\varepsilon|^m). \end{cases} \tag{3.1}$$

By (2.8) and the definition of

$$\widehat{\varphi}_1(x) = \varepsilon^{\frac{N}{q+1}} \varphi_1(\varepsilon x) \quad \text{and} \quad \widehat{\varphi}_2(x) = \varepsilon^{\frac{N}{p+1}} \varphi_2(\varepsilon x),$$

we indeed have

$$\begin{cases} W_1 = U_{\xi,\frac{1}{\Lambda}} - \widehat{\varphi}_1 = U_{\xi,\frac{1}{\Lambda}} - \varepsilon(\Lambda)^{1-\frac{N}{q+1}}\varphi_{1,0}\Big(\dfrac{x-\xi}{\Lambda}\Big) + \mathcal{O}(\varepsilon^2|\ln\varepsilon|^m), \\ W_2 = V_{\xi,\frac{1}{\Lambda}} - \widehat{\varphi}_2 = V_{\xi,\frac{1}{\Lambda}} - \varepsilon(\Lambda)^{1-\frac{N}{p+1}}\varphi_{2,0}\Big(\dfrac{x-\xi}{\Lambda}\Big) + \mathcal{O}(\varepsilon^2|\ln\varepsilon|^m). \end{cases} \tag{3.2}$$

Furthermore, we have the following upper bound

$$\begin{cases} |\widehat{\varphi}_1| \le \dfrac{\varepsilon|\ln\varepsilon|^n}{(1+|x-\xi|)^{N-3}}, \\ |\widehat{\varphi}_2| \le \dfrac{\varepsilon|\ln\varepsilon|^n}{(1+|x-\xi|)^{N-3}}, \end{cases} \tag{3.3}$$

and

$$\begin{cases} |W_1| \le C(U_{\xi,\frac{1}{\Lambda}})^{1-\tau}, \\ |W_2| \le C(V_{\xi,\frac{1}{\Lambda}})^{1-\tau}. \end{cases} \tag{3.4}$$

In the above, m,n,τ are chosen in Lemma A.5.

We define a weighted Sobolev norm as (1.9). For any function f in Ω_ε and $\xi \in \Omega_\varepsilon$, we define

$$||f||_{*,i} = ||f||_{W^{2,t_i}_{\gamma_i}(\Omega_\varepsilon)}, \tag{3.5a}$$

$$||f||_{**,i} = ||f||_{L^{t_i}_{\gamma_i}(\Omega_\varepsilon)}, \tag{3.5b}$$

where t_i, γ_i, $i=1,2$ are to be determined later.

For $t_i > N$, by Sobolev embedding theorem, we have

$$|\nabla f| + |f| \le C\langle x-\xi\rangle^{-\gamma_i}\|f\|_{*,i}, \quad i=1,2. \tag{3.6}$$

For any vector function (f_1,f_2), we define

$$\|(f_1,f_2)\|_* = \|f_1\|_{*,1} + \|f_2\|_{*,2},, \quad \|(f_1,f_2)\|_{**} = \|f_1\|_{**,1} + \|f_2\|_{**,2}.$$

Let τ_j be the j-th tangent on $\partial\Omega \cap B(a,\delta)$ with $1 \le j \le N-1$. For $i=1,2$, we define that

$$Y_0^i = \frac{\partial W_i}{\partial \Lambda}, \qquad Y_j^i = \frac{\partial W_i}{\partial \tau_j},$$

$$Z_0^i = -\Delta\frac{\partial W_i}{\partial \Lambda} + \mu\varepsilon^2\frac{\partial W_i}{\partial \Lambda}, \qquad Z_j^i = -\Delta\frac{\partial W_i}{\partial \tau_j} + \mu\varepsilon^2\frac{\partial W_i}{\partial \tau_j},$$

$$(u,v)_\varepsilon = \int_{\Omega_\varepsilon}(\nabla u\nabla v + \mu\varepsilon^2 uv).$$

3.1 Linear problem

We consider the following linear problem

$$\begin{cases} -\Delta\phi_1+\mu\varepsilon^2\phi_1-(p+\alpha\varepsilon)W_2^{p+\alpha\varepsilon-1}\phi_2=h_1+\sum\limits_{j=0}^{N-1}c_j^2Z_j^2 & \text{in } \Omega_\varepsilon,\\ -\Delta\phi_2+\mu\varepsilon^2\phi_2-(q+\beta\varepsilon)W_1^{q+\beta\varepsilon-1}\phi_1=h_2+\sum\limits_{j=0}^{N-1}c_j^1Z_j^1 & \text{in } \Omega_\varepsilon,\\ \dfrac{\partial\phi_1}{\partial n}=\dfrac{\partial\phi_2}{\partial n}=0 & \text{on } \partial\Omega_\varepsilon,\\ \langle Z_j^1,\phi_2\rangle=\langle Z_j^2,\phi_1\rangle=0, & j=0,\cdots,N-1, \end{cases} \tag{3.7}$$

where $\langle u,v\rangle=\int_{\Omega_\varepsilon}uv$, c_j^1, c_j^2 are some constants. Let T_ε be a linear operator that maps (h_1,h_2) in (3.7) to (ϕ_1,ϕ_2) in (3.7).

Lemma 3.1. *Assume $(\phi_{1,\varepsilon},\phi_{2,\varepsilon})$ solves* (3.7) *for $(h_1,h_2)=(h_{1,\varepsilon},h_{2,\varepsilon})$. If $\|(h_{1,\varepsilon},h_{2,\varepsilon})\|_{**}\to 0$ as $\varepsilon\to 0$, so does $\|(\phi_{1,\varepsilon},\phi_{2,\varepsilon})\|_*$.*

Proof. We argue by contradiction. Suppose that there are $n\to+\infty$ such that $\varepsilon_n\to 0$, $x_n\in\Omega_\varepsilon\cap B_\delta(\xi)$, $\frac{1}{\delta}<\Lambda_n<\delta$ and the solution $\|(\phi_{1,\varepsilon_n},\phi_{2,\varepsilon_n})\|_*\geq c>0$, where $\delta>0$ is a small constant independent of n. For simplicity, we drop the subscript n, and assume that $\|(\phi_{1,\varepsilon},\phi_{2,\varepsilon})\|_*=1$. Multiplying the first equation in (3.7) by Y_j^2, the second equation by Y_j^1 and integrating in Ω_ε, we find for $j=0,\cdots,N-1$ that

$$\begin{cases} \langle -\Delta Y_j^2+\mu\varepsilon^2Y_j^2,\phi_{1,\varepsilon}\rangle-\langle (p+\alpha\varepsilon)W_2^{p+\alpha\varepsilon-1}Y_j^2,\phi_{2,\varepsilon}\rangle=\langle h_{1,\varepsilon},Y_j^2\rangle+\sum\limits_{k=0}^{N-1}c_k^2\langle Z_k^2,Y_j^2\rangle,\\ \langle -\Delta Y_j^1+\mu\varepsilon^2Y_j^1,\phi_{2,\varepsilon}\rangle-\langle (q+\beta\varepsilon)W_1^{q+\beta\varepsilon-1}Y_j^1,\phi_{1,\varepsilon}\rangle=\langle h_{2,\varepsilon},Y_j^1\rangle+\sum\limits_{k=0}^{N-1}c_k^1\langle Z_k^1,Y_j^1\rangle. \end{cases} \tag{3.8}$$

On the one hand, we check, in view of the definition of Z_k^i, Y_j^i,

$$\begin{aligned} &\langle Z_0^i,Y_0^i\rangle=\|Y_0^i\|_\varepsilon^i=C_0^i+o(1),\\ &\langle Z_j^i,Y_j^i\rangle=\|Y_j^i\|_\varepsilon^i=C_1^i+o(1), \quad j=1,\cdots,N-1,\\ &\langle Z_k^i,Y_j^i\rangle=o(1), \qquad\qquad\qquad k\neq j, \end{aligned}$$

where C_0^i, C_1^i are strictly positive constants, $i=1,2$.

On the other hand,

$$\begin{aligned} &\langle -\Delta Y_j^2+\mu\varepsilon^2Y_j^2,\phi_{1,\varepsilon}\rangle=\langle -\Delta Y_j^1+\mu\varepsilon^2Y_j^1,\phi_{2,\varepsilon}\rangle=0,\\ &\langle (p+\alpha\varepsilon)W_2^{p+\alpha\varepsilon-1}Y_j^2,\phi_{2,\varepsilon}\rangle=o(\|\phi_{2,\varepsilon}\|_{*,2}),\\ &\langle (q+\beta\varepsilon)W_1^{q+\beta\varepsilon-1}Y_j^1,\phi_{1,\varepsilon}\rangle=o(\|\phi_{1,\varepsilon}\|_{*,1}),\\ &\langle h_{1,\varepsilon},Y_j^2\rangle=\mathcal{O}(\|h_{1,\varepsilon}\|_{**,1}), \quad \langle h_{2,\varepsilon},Y_j^1\rangle=\mathcal{O}(\|h_{2,\varepsilon}\|_{**,2}). \end{aligned}$$

Consequently, we have

$$c_j^1 = \mathcal{O}(\|h_{2,\varepsilon}\|_{**,2}) + o(\|\phi_{1,\varepsilon}\|_{*,1}), \quad c_j^2 = \mathcal{O}(\|h_{1,\varepsilon}\|_{**,1}) + o(\|\phi_{2,\varepsilon}\|_{*,2}).$$

In particular, $c_j^1 = o(1)$, $c_j^2 = o(1)$ as ε goes to zero.

Since $\|(\phi_{1,\varepsilon}, \phi_{2,\varepsilon})\|_* = 1$, the standard elliptic theory shows that as $\varepsilon \to 0$, $(\phi_{1,\varepsilon}(x-\xi), \phi_{2,\varepsilon}(x-\xi))$ converges uniformly in any compact set to some solution (ψ_1, ψ_2) to

$$\begin{cases} -\Delta\psi_1 - pV_{0,\widetilde{\Lambda}}^{p-1}\psi_2 = 0 & \text{in } \mathbb{R}_+^N, \\ -\Delta\psi_2 - qU_{0,\widetilde{\Lambda}}^{q-1}\psi_1 = 0 & \text{in } \mathbb{R}_+^N, \end{cases} \tag{3.9}$$

for some $\widetilde{\Lambda} > 0$. As a consequence, recalling (2.8),

$$\psi_1 = \widetilde{c}_0^1 \frac{\partial U_{0,\widetilde{\Lambda}}}{\partial \widetilde{\Lambda}} + \sum_{j=1}^{N-1} \widetilde{c}_j^1 \frac{\partial U_{0,\widetilde{\Lambda}}}{\partial a_j},$$

$$\psi_2 = \widetilde{c}_0^2 \frac{\partial V_{0,\widetilde{\Lambda}}}{\partial \widetilde{\Lambda}} + \sum_{j=1}^{N-1} \widetilde{c}_j^2 \frac{\partial V_{0,\widetilde{\Lambda}}}{\partial a_j}.$$

On the other hand, by orthogonality, we have

$$\int_{\mathbb{R}_+^N} -\Delta \frac{\partial U_{0,\widetilde{\Lambda}}}{\partial \widetilde{\Lambda}} \psi_2 = \int_{\mathbb{R}_+^N} U_{0,\widetilde{\Lambda}}^{q-1} \frac{\partial U_{0,\widetilde{\Lambda}}}{\partial \widetilde{\Lambda}} \psi_1 = 0,$$

$$\int_{\mathbb{R}_+^N} -\Delta \frac{\partial V_{0,\widetilde{\Lambda}}}{\partial \widetilde{\Lambda}} \psi_1 = \int_{\mathbb{R}_+^N} V_{0,\widetilde{\Lambda}}^{p-1} \frac{\partial V_{0,\widetilde{\Lambda}}}{\partial \widetilde{\Lambda}} \psi_2 = 0,$$

$$\int_{\mathbb{R}_+^N} -\Delta \frac{\partial U_{0,\widetilde{\Lambda}}}{\partial a_j} \psi_2 = \int_{\mathbb{R}_+^N} U_{0,\widetilde{\Lambda}}^{q-1} \frac{\partial U_{0,\widetilde{\Lambda}}}{\partial a_j} \psi_1 = 0,$$

$$\int_{\mathbb{R}_+^N} -\Delta \frac{\partial V_{0,\widetilde{\Lambda}}}{\partial a_j} \psi_1 = \int_{\mathbb{R}_+^N} V_{0,\widetilde{\Lambda}}^{p-1} \frac{\partial V_{0,\widetilde{\Lambda}}}{\partial a_j} \psi_2 = 0.$$

Moreover, there holds that

$$\int_{\mathbb{R}_+^N} \nabla \frac{\partial U_{0,\widetilde{\Lambda}}}{\partial \widetilde{\Lambda}} \nabla \frac{\partial U_{0,\widetilde{\Lambda}}}{\partial a_j} = \int_{\mathbb{R}_+^N} \nabla \frac{\partial U_{0,\widetilde{\Lambda}}}{\partial a_j} \nabla \frac{\partial U_{0,\widetilde{\Lambda}}}{\partial a_k} = 0, \qquad j \neq k,$$

$$\int_{\mathbb{R}_+^N} \nabla \frac{\partial V_{0,\widetilde{\Lambda}}}{\partial \widetilde{\Lambda}} \nabla \frac{\partial V_{0,\widetilde{\Lambda}}}{\partial a_j} = \int_{\mathbb{R}_+^N} \nabla \frac{\partial V_{0,\widetilde{\Lambda}}}{\partial a_j} \nabla \frac{\partial V_{0,\widetilde{\Lambda}}}{\partial a_k} = 0, \qquad j \neq k,$$

$$\int_{\mathbb{R}_+^N} |\nabla \frac{\partial U_{0,\widetilde{\Lambda}}}{\partial \widetilde{\Lambda}}|^2 = \widetilde{C}_0^1 > 0, \quad \int_{\mathbb{R}_+^N} |\nabla \frac{\partial U_{0,\widetilde{\Lambda}}}{\partial a_j}|^2 = \widetilde{C}_1^1 > 0, \qquad 1 \leq j \leq N-1,$$

$$\int_{\mathbb{R}_+^N} |\nabla \frac{\partial V_{0,\widetilde{\Lambda}}}{\partial \widetilde{\Lambda}}|^2 = \widetilde{C}_0^2 > 0, \quad \int_{\mathbb{R}_+^N} |\nabla \frac{\partial V_{0,\widetilde{\Lambda}}}{\partial a_j}|^2 = \widetilde{C}_1^2 > 0, \qquad 1 \leq j \leq N-1.$$

Hence, $\widetilde{c}_j^1, \widetilde{c}_j^2$ solve a homogeneous quasi diagonal linear system, yielding $\widetilde{c}_j^1 = 0, \widetilde{c}_j^2 = 0$, $0 \le j \le N-1$, and $\psi_1 = 0$, $\psi_2 = 0$. So $\phi_{1,\varepsilon}(x-\xi) \to 0$, $\phi_{2,\varepsilon}(x-\xi) \to 0$ in $C^1_{loc}(\Omega_\varepsilon)$. By (3.4) and Remark 2.1, we have that

$$|\langle x-\xi\rangle^{\gamma_1+2} W_1^{q+\beta\varepsilon-1}\phi_{1,\varepsilon}|^{t_1} \le C\|\phi_{1,\varepsilon}\|_{*,1}^{t_1} \langle x-\xi\rangle^{(2-[N-2](q+\beta\varepsilon-1)(1-\tau_1))t_1} \in L^1(\mathbb{R}^N),$$
$$|\langle x-\xi\rangle^{\gamma_2+2} W_2^{p+\alpha\varepsilon-1}\phi_{2,\varepsilon}|^{t_2} \le C\|\phi_{2,\varepsilon}\|_{*,2}^{t_2} \langle x-\xi\rangle^{(2-[N-2](p+\alpha\varepsilon-1)(1-\tau_2))t_2} \in L^1(\mathbb{R}^N),$$

hold for $p > \frac{1}{N-2}(N+\frac{N}{t_2})$, which actually ensures that $(2-[N-2](p-1))t_2 < -N$. Then we obtain, by Dominated Convergence Theorem, that

$$\|(W_1^{q+\beta\varepsilon-1}\phi_{1,\varepsilon}, W_2^{p+\alpha\varepsilon-1}\phi_{2,\varepsilon})\|_{**} = o(1).$$

On the other hand, under the condition $p > \frac{1}{N-2}(N+\frac{N}{t_2})$, we know that

$$\begin{aligned}
&\langle x-\xi\rangle^{\gamma_1+2}|Z_j^1| \le C\langle x-\xi\rangle^{\gamma_1+2-(N-2)p} \in L^{t_1}(\mathbb{R}^N),\\
&\langle x-\xi\rangle^{\gamma_2+2}|Z_j^2| \le C\langle x-\xi\rangle^{\gamma_2+2-(N-2)q} \in L^{t_2}(\mathbb{R}^N),\\
&\|(\phi_{1,\varepsilon},\phi_{2,\varepsilon})\|_* \le C\|(W_1^{q+\beta\varepsilon-1}\phi_{1,\varepsilon}, W_2^{p+\alpha\varepsilon-1}\phi_{2,\varepsilon})\|_{**} + C\|(h_{1,\varepsilon},h_{2,\varepsilon})\|_{**}\\
&\qquad\qquad + C\sum_{j=0}^{N-1}(|c_j^1|\|Z_j^1\|_{**,1} + |c_j^2|\|Z_j^2\|_{**,2}) = o(1),
\end{aligned}$$

that is, a contradiction. □

As a result of Lemma 3.1, we can prove the following result.

Proposition 3.1. *There exist $\varepsilon_0 > 0$ and some constant $C > 0$, independent of ε and ξ, such that for all $0 < \varepsilon < \varepsilon_0$ and all $(h_{1,\varepsilon}, h_{2,\varepsilon}) \in L^{t_1}_{\gamma_1+2}(\Omega_\varepsilon) \times L^{t_2}_{\gamma_2+2}(\Omega_\varepsilon)$ satisfying the assumptions in Lemma 3.1, the linear problem* (3.7) *has a unique solution $(\phi_{1,\varepsilon},\phi_{2,\varepsilon}) \equiv T_\varepsilon(h_{1,\varepsilon},h_{2,\varepsilon})$. Moreover, there hold that*

$$\|T_\varepsilon(h_{1,\varepsilon},h_{2,\varepsilon})\|_* \le C\|(h_{1,\varepsilon},h_{2,\varepsilon})\|_{**}, \quad |c_j^i| \le C\|(h_{1,\varepsilon},h_{2,\varepsilon})\|_{**}.$$

3.2 Nonlinear problem

We aim to solve the nonlinear problem

$$\begin{cases}
-\Delta(W_1+\phi_1) + \mu\varepsilon^2(W_1+\phi_1) = (W_2+\phi_2)^{p+\alpha\varepsilon} + \sum_{k=0}^{N-1} c_k^2 Z_k^2,\\
-\Delta(W_2+\phi_2) + \mu\varepsilon^2(W_2+\phi_2) = (W_1+\phi_1)^{q+\beta\varepsilon} + \sum_{k=0}^{N-1} c_k^1 Z_k^1,\\
(\phi_1,\phi_2) \in W^{2,t_1}_{\gamma_1}(\Omega_\varepsilon) \times W^{2,t_2}_{\gamma_2}(\Omega_\varepsilon),\\
\langle Z_k^1, \phi_2\rangle = \langle Z_k^2, \phi_1\rangle = 0, \quad k = 1,\cdots,N-1.
\end{cases} \tag{3.10}$$

Rewrite (3.10) as

$$\begin{cases} L_\varepsilon(\phi_1,\phi_2) = R_\varepsilon + N_\varepsilon(\phi_1,\phi_2) + \sum\limits_{k=0}^{N-1}(c_k^2 Z_k^2, c_k^1 Z_k^1), \\ (\phi_1,\phi_2) \in W^{2,t_1}_{\gamma_1}(\Omega_\varepsilon) \times W^{2,t_2}_{\gamma_2}(\Omega_\varepsilon), \\ \langle Z_k^1, \phi_2\rangle = \langle Z_k^2, \phi_1\rangle = 0, \quad k=1,\cdots,N-1, \end{cases} \tag{3.11}$$

where

$$\begin{aligned} L_\varepsilon(\phi_1,\phi_2) = &(-\Delta\phi_1 + \mu\varepsilon^2\phi_1 - (p+\alpha\varepsilon)W_2^{p+\alpha\varepsilon-1}\phi_2, -\Delta\phi_2 \\ &+ \mu\varepsilon^2\phi_2 - (q+\beta\varepsilon)W_1^{q+\beta\varepsilon-1}\phi_1), \end{aligned} \tag{3.12a}$$

$$\begin{aligned} R_\varepsilon = (R_{\varepsilon,1}, R_{\varepsilon,2}) &= (\Delta W_1 - \mu\varepsilon^2 W_1 + W_2^{p+\alpha\varepsilon}, \Delta W_2 - \mu\varepsilon^2 W_2 + W_1^{q+\beta\varepsilon}) \\ &= (W_2^{p+\alpha\varepsilon} - V^p_{\xi,\frac{1}{\Lambda}}, W_1^{q+\beta\varepsilon} - U^q_{\xi,\frac{1}{\Lambda}}), \end{aligned} \tag{3.12b}$$

$$N_\varepsilon(\phi_1,\phi_2) = \Big(N_{\varepsilon,1}(\phi_2), N_{\varepsilon,2}(\phi_1)\Big), \tag{3.12c}$$

with

$$\begin{aligned} N_{\varepsilon,1}(\phi_2) &= (W_2+\phi_2)^{p+\alpha\varepsilon} - W_2^{p+\alpha\varepsilon} - (p+\alpha\varepsilon)W_2^{p+\alpha\varepsilon-1}\phi_2, \\ N_{\varepsilon,2}(\phi_1) &= (W_1+\phi_1)^{q+\beta\varepsilon} - W_1^{q+\beta\varepsilon} - (q+\beta\varepsilon)W_1^{q+\beta\varepsilon-1}\phi_1. \end{aligned}$$

In order to use the contraction mapping theorem, we estimate the error term R_ε and the higher order term $N_\varepsilon(\phi_1,\phi_2)$.

Lemma 3.2. *Under the assumption* (A)*, there exists a constant* $C>0$*, independent of* ξ *such that*

$$\|R_\varepsilon\|_{**} \le C\varepsilon, \tag{3.13a}$$

$$\|D_{(\Lambda,\xi)} R_\varepsilon\|_{**} \le C\varepsilon. \tag{3.13b}$$

Proof. First, we show in Appendix A that

$$\begin{aligned} W_1 &= U_{\xi,\frac{1}{\Lambda}} + \mathcal{O}\big(\varepsilon U_{\xi,\frac{1}{\Lambda}}^{\frac{N-3}{N-2}(1-\tau)}\big), \\ W_2 &= V_{\xi,\frac{1}{\Lambda}} + \mathcal{O}\big(\varepsilon V_{\xi,\frac{1}{\Lambda}}^{\frac{N-3}{N-2}(1-\tau)}\big), \end{aligned}$$

where $\tau>0$ is as small as desired. Recalling $p\le q$ as we set, by (3.12b), we obtain

$$\begin{aligned} R_{\varepsilon,1} =& W_2^{p+\alpha\varepsilon} - V^p_{\xi,\frac{1}{\Lambda}} \\ =& W_2^{p+\alpha\varepsilon} - W_2^p + W_2^p - V^p_{\xi,\frac{1}{\Lambda}} \\ =& \mathcal{O}\Big(\alpha\varepsilon W_2^p|\ln W_2| + \varepsilon V_{\xi,\frac{1}{\Lambda}}^{\frac{N-3}{N-2}(1-\tau)+p-1}\Big) \\ =& \mathcal{O}\Big(\varepsilon V_{\xi,\frac{1}{\Lambda}}^{p(1-\tau)}|\ln V_{\xi,\frac{1}{\Lambda}}| + \varepsilon V_{\xi,\frac{1}{\Lambda}}^{\frac{N-3}{N-2}(1-\tau)+p-1}\Big), \end{aligned}$$

and

$$
\begin{aligned}
R_{\varepsilon,2} =& W_1^{q+\beta\varepsilon} - U^q_{\xi,\frac{1}{\Lambda}} \\
=& W_1^{q+\beta\varepsilon} - W_1^q + W_1^q - U^q_{\xi,\frac{1}{\Lambda}} \\
=& \mathcal{O}\Big(\varepsilon U^{q(1-\tau)}_{\xi,\frac{1}{\Lambda}} |\ln U_{\xi,\frac{1}{\Lambda}}| + \varepsilon U^{\frac{N-3}{N-2}(1-\tau)+q-1}_{\xi,\frac{1}{\Lambda}}\Big).
\end{aligned}
$$

Then

$$
\|R_{\varepsilon,1}\|_{**,2} \le C\varepsilon \|\langle x-\xi\rangle^{\gamma_2+2}\Big(V^{p(1-\tau)}_{\xi,\frac{1}{\Lambda}}|\ln V_{\xi,\frac{1}{\Lambda}}| + V^{\frac{N-3}{N-2}(1-\tau)+p-1}_{\xi,\frac{1}{\Lambda}}\Big)\|_{L^{t_2}(\Omega_\varepsilon)} \le C\varepsilon, \tag{3.14}
$$

and

$$
\|R_{\varepsilon,2}\|_{**,1} \le C\varepsilon \|\langle x-\xi\rangle^{\gamma_1+2}\Big(U^{q(1-\tau)}_{\xi,\frac{1}{\Lambda}}|\ln U_{\xi,\frac{1}{\Lambda}}| + U^{\frac{N-3}{N-2}(1-\tau)+q-1}_{\xi,\frac{1}{\Lambda}}\Big)\|_{L^{t_1}(\Omega_\varepsilon)} \le C\varepsilon.
$$

Differentiating (3.12b) with respect to the parameter Λ and ξ, respectively. Then the estimate (3.13b) can be obtained in the same way. □

Remark 3.1. The assumption (A) here is sufficient. Note that the condition $p > \frac{1}{N-2}(N+\frac{N}{t_2})$ is necessary for the solvability of linear problem (see Lemma 3.2). With (3.14), we then derive that $\gamma_2 < N-3$.

Lemma 3.3. *Under the assumption* (A)*, there exist $\varepsilon_1 > 0$, independent of Λ, ξ, and C, independent of ε, Λ, ξ, such that for $|\varepsilon| \le \varepsilon_1$, and $\|(\phi_1,\phi_2)\|_* \le 1$,*

$$
\begin{aligned}
&\|N_{\varepsilon,1}(\phi_2)\|_{**,2} \le C\|\phi_2\|_{*,2}^{\min\{p+\alpha\varepsilon,2\}}, \\
&\|N_{\varepsilon,2}(\phi_1)\|_{**,1} \le C\|\phi_1\|_{*,1}^{\min\{q+\beta\varepsilon,2\}},
\end{aligned}
$$

and, for $\|(\phi_1,\phi_2)\|_* \le 1$ and $\|(\psi_1,\psi_2)\|_* \le 1$,

$$
\begin{aligned}
&\|N_{\varepsilon,1}(\phi_2) - N_{\varepsilon,1}(\psi_2)\|_{**,2} \le C(\max(\|\phi_2\|_{*,2},\|\psi_2\|_{*,2}))^{\min\{p+\alpha\varepsilon-1,1\}}\|\phi_2-\psi_2\|_{*,2}, \\
&\|N_{\varepsilon,2}(\phi_1) - N_{\varepsilon,2}(\psi_1)\|_{**,1} \le C(\max(\|\phi_1\|_{*,1},\|\psi_1\|_{*,1}))^{\min\{q+\beta\varepsilon-1,1\}}\|\phi_1-\psi_1\|_{*,1}.
\end{aligned}
$$

Proof. By definition of (3.12c), we have

$$
\begin{aligned}
&|N_{\varepsilon,1}(\phi_2)| \le \begin{cases} C|\phi_2|^{p+\alpha\varepsilon}, & \text{if } 1<p\le 2, \\ CW_2^{p+\alpha\varepsilon-2}|\phi_2|^2 + C|\phi_2|^{p+\alpha\varepsilon}, & \text{if } p>2, \end{cases} \\
&|N_{\varepsilon,2}(\phi_1)| \le \begin{cases} C|\phi_1|^{q+\beta\varepsilon}, & \text{if } 1<q\le 2, \\ CW_1^{q+\beta\varepsilon-2}|\phi_1|^2 + C|\phi_1|^{q+\beta\varepsilon}, & \text{if } q>2. \end{cases}
\end{aligned}
$$

We first estimate $N_{\varepsilon,1}(\phi_2)$. For $1<p\le 2$, once

$$
\gamma_2 > \frac{2+\frac{N}{t_2}}{p-1} \tag{3.15}
$$

holds, we have

$$\begin{aligned}\||\phi_2|^{p+\alpha\varepsilon}\|_{**,2} =&\Big(\int_{\Omega_\varepsilon}(\langle x-\xi\rangle^{\gamma_2+2}|\phi_2|^{p+\alpha\varepsilon})^{t_2}\Big)^{\frac{1}{t_2}}\\ \le& C\|\phi_2\|_{*,2}^{p+\alpha\varepsilon}\Big(\int_{\Omega_\varepsilon}(\langle x-\xi\rangle^{\gamma_2+2-(p+\alpha\varepsilon)\gamma_2})^{t_2}\Big)^{\frac{1}{t_2}}\\ \le& C\|\phi_2\|_{*,2}^{p+\alpha\varepsilon}.\end{aligned}$$

Note that the assumption (A) ensures that (3.15) holds.

For $p>2$, noticing that W_2^ε is bounded since W_2 is bounded, we have

$$\begin{aligned}\|W_2^{p+\alpha\varepsilon-2}|\phi_2|^2\|_{**,2} =&\Big[\int_{\Omega_\varepsilon}(\langle x-\xi\rangle^{\gamma_2+2}W_2^{p+\alpha\varepsilon-2}|\phi_2|^2)^{t_2}\Big]^{\frac{1}{t_2}}\\ \le& C\|\phi_2\|_{*,2}^{2}\Big[\int_{\Omega_\varepsilon}(\langle x-\xi\rangle^{2-\gamma_2-(N-2)(p-2)(1-\tau_2)})^{t_2}\Big]^{\frac{1}{t_2}}\\ \le& C\|\phi_2\|_{*,2}^{2}.\end{aligned}$$

Thus, we obtain

$$\|N_{\varepsilon,1}(\phi_2)\|_{**,2}\le C\|\phi_2\|_{*,2}^{\min\{p+\alpha\varepsilon,2\}}.$$

Secondly, we estimate $N_{\varepsilon,2}(\phi_1)$. By the same way, we have

$$\|N_{\varepsilon,2}(\phi_1)\|_{**,1}\le C\|\phi_1\|_{*,1}^{\min\{q+\beta\varepsilon,2\}}$$

holds for $t_1>N$ and $\frac{N-2}{2}+\frac{N(N-2)}{4t_1}<\gamma_1<N-2-\frac{N}{t_1}$.

Last, we estimate $N_{\varepsilon,1}(\phi_2)-N_{\varepsilon,1}(\psi_2)$ and $N_{\varepsilon,2}(\phi_1)-N_{\varepsilon,1}(\psi_1)$. We write

$$\begin{cases} N_{\varepsilon,1}(\phi_2)-N_{\varepsilon,1}(\psi_2)=N'_{\varepsilon,1}(t\phi_2+(1-t)\psi_2)(\phi_2-\psi_2),\\ N_{\varepsilon,2}(\phi_1)-N_{\varepsilon,1}(\psi_1)=N'_{\varepsilon,1}(t\phi_1+(1-t)\psi_1)(\phi_1-\psi_1),\end{cases}$$

for some $t\in[0,1]$. Since, for any $\eta\in W^{2,t_1}_{\gamma_1}(\Omega_\varepsilon)$,

$$\begin{cases} N'_{\varepsilon,1}(\eta)=(p+\alpha\varepsilon)[(W_2+\eta)^{p+\alpha\varepsilon-1}-W_2^{p+\alpha\varepsilon-1}],\\ N'_{\varepsilon,2}(\eta)=(q+\beta\varepsilon)[(W_1+\eta)^{q+\beta\varepsilon-1}-W_1^{q+\beta\varepsilon-1}],\end{cases}$$

we can conclude the proof by repeating the previous steps. □

Remark 3.2. The assumption (A) can ensure that (3.15) holds. For the range of γ_2 in Remark 3.1, (3.15) forces that $\frac{1}{p-1}(2+\frac{N}{t_2})<N-3$, that is $p>\frac{1}{N-3}(N-1+\frac{N}{t_2})$. So we impose $N\ge5$ and $t_2>N(2-\frac{4}{N-2})^{-1}$ such that

$$\frac{N-1+\frac{N}{t_2}}{N-3}<\frac{N+2}{N-2}.$$

Therefore, we can give assumptions on N, p, γ_2 and t_2 that

$$N \geq 5, \quad p \in \left(\frac{N-1+\frac{N}{t_2}}{N-3}, \frac{N+2}{N-2}\right), \quad \frac{2+\frac{N}{t_2}}{p-1} < \gamma_2 < N-3, \quad t_2 > \max\left\{N, \frac{N}{2-\frac{4}{N-2}}\right\}.$$

Using Lemmas 3.2, 3.3 and the contraction mapping theorem, we obtain Proposition 3.2.

Proposition 3.2. *There exists some C, independent of ε and ξ, such that for small ε, the problem* (3.11) *has a unique solution $(\phi_1,\phi_2) = (\phi_1,\phi_2)(\Lambda,\xi,\mu,\varepsilon)$ satisfying*

$$\|(\phi_1,\phi_2)\|_* \leq C\varepsilon.$$

Moreover, $(\Lambda,\xi) \to (\phi_1,\phi_2)(\Lambda,\xi,\mu,\varepsilon)$ is C^1 with respect to the $W^{2,t_1}_{\gamma_1}(\Omega_\varepsilon) \times W^{2,t_2}_{\gamma_2}(\Omega_\varepsilon)$-norm, and

$$\|D_{(\Lambda,\xi)}(\phi_1,\phi_2)\|_* \leq C\varepsilon.$$

Proof. Define

$$\mathcal{F} = \{(\phi_1,\phi_2) \in W^{2,t_1}_{\gamma_1}(\Omega_\varepsilon) \times W^{2,t_2}_{\gamma_2}(\Omega_\varepsilon) : \|(\phi_1,\phi_2)\|_* \leq C_0\varepsilon\},$$

and

$$A_\varepsilon : \mathcal{F} \to W^{2,t_1}_{\gamma_1}(\Omega_\varepsilon) \times W^{2,t_2}_{\gamma_2}(\Omega_\varepsilon), \quad A_\varepsilon(\phi_1,\phi_2) = T_\varepsilon(N_\varepsilon(\phi_1,\phi_2) + R_\varepsilon),$$

where $C_0 > 0$ is some large positive constant independent of ε and ξ, and T_ε is defined as Proposition 3.1.

We claim that the operator A_ε have a fixed point. For this purpose, we prove that A_ε is a contraction mapping, then we finish the proof by the contraction mapping theorem. On the one hand, for $(\phi_1,\phi_2) \in \mathcal{F}$ and ε small enough, by Lemmas 3.2 and 3.3, we have

$$\|A_\varepsilon(\phi_1,\phi_2)\|_* = \|T_\varepsilon(N_\varepsilon(\phi_1,\phi_2) + R_\varepsilon)\|_* \leq C\|N_\varepsilon(\phi_1,\phi_2)\|_{**} + C\|R_\varepsilon\|_{**} \leq C_0\varepsilon,$$

where $C > 0$ is a constant defined in Proposition 3.7, and the last inequality holds for choosing C_0 large enough. Then A_ε maps $\mathcal{F}$ onto itself. On the other hand, for $(\phi_1,\phi_2) \in \mathcal{F}$, $(\psi_1,\psi_2) \in \mathcal{F}$ and ε sufficiently small

$$\begin{aligned}
&\|A_\varepsilon(\phi_1,\phi_2) - A_\varepsilon(\psi_1,\psi_2)\|_* \\
\leq& C\|N_\varepsilon(\phi_1,\phi_2) - N_\varepsilon(\psi_1,\psi_2)\|_{**} \\
\leq& C[\varepsilon^{\min\{p+\alpha\varepsilon-1,1\}}\|\phi_2 - \psi_2\|_{*,2} + \varepsilon^{\min\{q+\beta\varepsilon-1,1\}}\|\phi_1 - \psi_1\|_{*,1}] \\
\leq& \frac{1}{2}\|(\phi_1,\phi_2) - (\psi_1,\psi_2)\|_*,
\end{aligned}$$

that is A_ε is a contraction mapping.

Therefore, the problem (3.11) has a unique solution (ϕ_1,ϕ_2) satisfying $\|(\phi_1,\phi_2)\|_* \le C_0\varepsilon$.

Next, we show that the mapping $(\Lambda,\xi)\to(\phi_1,\phi_2)(\Lambda,\xi)$ is C^1. We define

$$B(\Lambda,\xi,\eta_1,\eta_2):=(\eta_1,\eta_2)-T_\varepsilon(N_\varepsilon(\eta_1,\eta_2)+R_\varepsilon),$$

where Λ satisfying (2.7), $\xi\in\Omega_\varepsilon$ and $(\eta_1,\eta_2)\in\mathcal{F}$. Obviously, $B(\Lambda,\xi,\eta_1,\eta_2)$ is C^1 with respect to Λ, ξ, η_1, and η_2, and (ϕ_1,ϕ_2) satisfies

$$B(\Lambda,\xi,\phi_1,\phi_2)=0. \tag{3.16}$$

In order to use the implicit function theorem, we calculate

$$\partial_{(\eta_1,\eta_2)}B(\Lambda,\xi,\eta_1,\eta_2)[(\theta_1,\theta_2)]=(\theta_1,\theta_2)-T_\varepsilon((\theta_1,\theta_2)(\partial_{(\eta_1,\eta_2)}N_\varepsilon)(\eta_1,\eta_2)).$$

For $p>2$ and $q>2$, using Proposition 3.1, (3.6) and Lemma 3.3, we obtain

$$\begin{aligned}
&\|T_\varepsilon((\theta_1,\theta_2)(\partial_{(\eta_1,\eta_2)}N_\varepsilon)(\eta_1,\eta_2))\|_*\\
\le& C\|(\theta_1,\theta_2)(\partial_{(\eta_1,\eta_2)}N_\varepsilon)(\eta_1,\eta_2)\|_{**}\\
\le& C[\|\langle x-\xi\rangle^{-\gamma_2}(\partial_{\eta_2}N_{\varepsilon,1})(\eta_2)\|_{**,2}\|\theta_2\|_{*,2}+\|\langle x-\xi\rangle^{-\gamma_1}(\partial_{\eta_1}N_{\varepsilon,2})(\eta_1)\|_{**,1}\|\theta_1\|_{*,1}]\\
\le& C[\|\langle x-\xi\rangle^2|\eta_2|^{p+\alpha\varepsilon-1}\|_{L^{t_2}(\Omega_\varepsilon)}\|\theta_2\|_{*,2}+\|\langle x-\xi\rangle^2|\eta_1|^{q+\beta\varepsilon-1}\|_{L^{t_1}(\Omega_\varepsilon)}\|\theta_1\|_{*,1}]\\
\le& C[C(\gamma_2)\|\eta_2\|_{*,2}^{p+\alpha\varepsilon-1}\|\theta_2\|_{*,2}+C(\gamma_1)\|\eta_1\|_{*,1}^{q+\beta\varepsilon-1}\|\theta_1\|_{*,1}]\\
\le& C\varepsilon^{\sigma_1}\|(\theta_1,\theta_2)\|_*,
\end{aligned}$$

where $\sigma_1=\sigma_1(p,q)$ is a constant. Similarly, for $(p,q)\in\{p>2,\ q>2\}^c$, there exists $\sigma_2=\sigma_2(p,q)$ such that

$$\|T_\varepsilon((\theta_1,\theta_2)(\partial_{(\eta_1,\eta_2)}N_\varepsilon)(\eta_1,\eta_2))\|_*\le C\varepsilon^{\sigma_2}\|(\theta_1,\theta_2)\|_*.$$

Thus

$$\|T_\varepsilon((\theta_1,\theta_2)(\partial_{(\eta_1,\eta_2)}N_\varepsilon)(\eta_1,\eta_2))\|_*\le C\varepsilon^{\min(\sigma_1,\sigma_2)}\|(\theta_1,\theta_2)\|_*.$$

Therefore, $\partial_{(\eta_1,\eta_2)}B(\Lambda,\xi,\phi_1,\phi_2)$ is invertible in $W^{2,t_1}_{\gamma_1}(\Omega_\varepsilon)\times W^{2,t_2}_{\gamma_2}(\Omega_\varepsilon)$ with uniformly bounded inverse. Then $(\Lambda,\xi)\to(\phi_1,\phi_2)(\Lambda,\xi)$ is C^1 follows from the implicit functions theorem.

Differentiating (3.16) with respect to Λ, we obtain

$$\begin{aligned}
\partial_\Lambda(\phi_1,\phi_2)=&(\partial_{(\eta_1,\eta_2)}B(\Lambda,\xi,\phi_1,\phi_2))^{-1}(\partial_\Lambda T_\varepsilon)(N_\varepsilon)(\phi_1,\phi_2)\\
&+T_\varepsilon((\partial_\Lambda N_\varepsilon)(\phi_1,\phi_2))+\partial_\Lambda(T_\varepsilon(R_\varepsilon)),
\end{aligned}$$

then, by Proposition 3.7,

$$\begin{aligned}
\|\partial_\Lambda(\phi_1,\phi_2)\|_*=&\|(\partial_\Lambda T_\varepsilon)(N_\varepsilon)(\phi_1,\phi_2)\|_*+\|T_\varepsilon((\partial_\Lambda N_\varepsilon)(\phi_1,\phi_2))\|_*+\|\partial_\Lambda(T_\varepsilon(R_\varepsilon))\|_*\\
\le& C(\|N_\varepsilon(\phi_1,\phi_2)\|_{**}+\|(\partial_\Lambda N_\varepsilon)(\phi_1,\phi_2)\|_{**}+\|R_\varepsilon\|_{**}).
\end{aligned}$$

By the boundedness of W_1^ε, W_2^ε, we have

$$\begin{aligned}
&\|(\partial_\Lambda N_\varepsilon)(\phi_1,\phi_2)\|_{**}\\
=&\|(\partial_\Lambda N_{\varepsilon,1})(\phi_2)\|_{**,2}+\|(\partial_\Lambda N_{\varepsilon,2})(\phi_1)\|_{**,1}\\
=&\|(p+\alpha\varepsilon)|(W_2+\phi_2)^{p+\alpha\varepsilon-1}-W_2^{p+\alpha\varepsilon-1}-(p+\alpha\varepsilon-1)W_2^{p+\alpha\varepsilon-2}\phi_2||\partial_\Lambda W_2|\|_{**,2}\\
&\quad+\|(q+\beta\varepsilon)|(W_1+\phi_1)^{q+\beta\varepsilon-1}-W_1^{q+\beta\varepsilon-1}-(q+\beta\varepsilon-1)W_1^{q+\beta\varepsilon-2}\phi_1||\partial_\Lambda W_1|\|_{**,1}\\
\le& C\begin{cases}\|W_2|\phi_2|^{p+\alpha\varepsilon-1}+W_2^{p+\alpha\varepsilon-1}|\phi_2|\|_{**,2}, & 1<p\le 2,\\ \|W_2|\phi_2|^{p+\alpha\varepsilon-1}\|_{**,2}, & 2<p\le 3,\\ \|W_2^{p+\alpha\varepsilon-2}|\phi_2|^2+W_2|\phi_2|^{p+\alpha\varepsilon-1}\|_{**,2}, & p>3,\end{cases}\\
&\quad+C\begin{cases}\|W_1|\phi_1|^{q+\beta\varepsilon-1}+W_1^{q+\beta\varepsilon-1}|\phi_1|\|_{**,2}, & 1<q\le 2,\\ \|W_1|\phi_1|^{q+\beta\varepsilon-1}\|_{**,2}, & 2<q\le 3,\\ \|W_1^{q+\beta\varepsilon-2}|\phi_2|^2+W_1|\phi_1|^{q+\beta\varepsilon-1}\|_{**,2}, & q>3,\end{cases}\\
\le& C\begin{cases}\|\phi_2\|_{*,2}^{p+\alpha\varepsilon-1}\|\langle x-\xi\rangle^{-(N-2)(1-\tau_2)-\gamma_2(p+\alpha\varepsilon-1)}\|_{**,2}\\ \quad+\|\phi_2\|_{*,2}\|\langle x-\xi\rangle^{-(N-2)(p+\alpha\varepsilon-1)(1-\tau_2)-\gamma_2}\|_{**,2}, & 1<p\le 2,\\ \|\phi_2\|_{*,2}^{p+\alpha\varepsilon-1}\|\langle x-\xi\rangle^{-(N-2)(1-\tau_2)-\gamma_2(p+\alpha\varepsilon-1)}\|_{**,2}, & 2<p\le 3,\\ \|\phi_2\|_{*,2}^2\|\langle x-\xi\rangle^{-(N-2)(p+\alpha\varepsilon-2)(1-\tau_2)-2\gamma_2}\|_{**,2}\\ \quad+\|\phi_2\|_{*,2}^{p+\alpha\varepsilon-1}\|\langle x-\xi\rangle^{-(N-2)(1-\tau_2)-\gamma_2(p+\alpha\varepsilon-1)}\|_{**,2}, & p>3,\end{cases}\\
&\quad+C\begin{cases}\|\phi_1\|_{*,1}^{q+\beta\varepsilon-1}\|\langle x-\xi\rangle^{-(N-2)(1-\tau_1)-\gamma_1(q+\beta\varepsilon-1)}\|_{**,1}\\ \quad+\|\phi_1\|_{*,1}\|\langle x-\xi\rangle^{-(N-2)(q+\beta\varepsilon-1)(1-\tau_1)-\gamma_1}\|_{**,1}, & 1<q\le 2,\\ \|\phi_1\|_{*,1}^{q+\beta\varepsilon-1}\|\langle x-\xi\rangle^{-(N-2)(1-\tau_1)-\gamma_1(q+\beta\varepsilon-1)}\|_{**,1}, & 2<q\le 3,\\ \|\phi_1\|_{*,1}^2\|\langle x-\xi\rangle^{-(N-2)(q+\beta\varepsilon-2)(1-\tau_1)-2\gamma_1}\|_{**,1}\\ \quad+\|\phi_1\|_{*,1}^{q+\beta\varepsilon-1}\|\langle x-\xi\rangle^{-(N-2)(1-\tau_1)-\gamma_1(q+\beta\varepsilon-1)}\|_{**,1}, & q>3,\end{cases}\\
\le& C\varepsilon.
\end{aligned}$$

Combined with Lemmas 3.2 and 3.3, we deduce $\|\partial_\Lambda(\phi_1,\phi_2)\|_*\le C\varepsilon$. Similarly, we can obtain $\|\partial_\xi(\phi_1,\phi_2)\|_*\le C\varepsilon$. This completes the proof. □

4 The reduced problem

We introduce the following functional defined in $(W^{2,\frac{p+1}{p}}\cap W^{1,p^*})\times(W^{2,\frac{q+1}{q}}\cap W^{1,q^*})$

$$\begin{aligned}
J_\varepsilon(u,v):=&\int_{\Omega_\varepsilon}\nabla u_1\cdot\nabla u_2+\mu\varepsilon^2\int_{\Omega_\varepsilon}u_1u_2\\
&-\frac{1}{p+\alpha\varepsilon+1}\int_{\Omega_\varepsilon}|u_2|^{p+\alpha\varepsilon+1}-\frac{1}{q+\beta\varepsilon+1}\int_{\Omega_\varepsilon}|u_1|^{q+\beta\varepsilon+1},
\end{aligned}$$

whose, by standard arguments, nontrivial critical points are solutions to (1.4).

Setting

$$I_\varepsilon(\Lambda,a)\equiv J_\varepsilon(W_1+\phi_{1,\varepsilon},W_2+\phi_{2,\varepsilon}).$$

We are reduced to prove the following result.

Proposition 4.1. *The function* $(W_1+\phi_1,W_2+\phi_2)$ *is a solution to problem* (1.4) *if and only if* (Λ,a) *is a critical point of* I_ε.

Proposition 4.2. *There exist* A_1, A_2, B_1, B_2, C_1, C_2, D_1, D_2, *strictly positive constants such that*

$$\begin{aligned}I_\varepsilon(\Lambda,a)=&\Big(\frac{q-1}{2}A_1+\frac{p-1}{2}A_2\Big)-\varepsilon\Lambda H(a)(B_1+B_2+C_1+C_2)\\&+\varepsilon\ln\Lambda\Big(\frac{N\beta}{q+1}A_1+\frac{N\alpha}{p+1}A_2\Big)+\varepsilon\Big(\frac{\beta}{2}A_1+\frac{\alpha}{2}A_2+\beta D_1+\alpha D_2\Big)+\varepsilon\sigma_\varepsilon(\Lambda,a),\end{aligned}$$

with σ_ε *and* $\partial_\Lambda\sigma_\varepsilon$ *going to zero as* ε *goes to zero, uniformly with respect to* Λ *satisfying* (2.7).

Proof. We show in Appendix B that

$$\begin{aligned}J_\varepsilon(W_1,W_2)=&\Big(\frac{q-1}{2}A_1+\frac{p-1}{2}A_2\Big)-\varepsilon\Lambda H(a)(B_1+B_2+C_1+C_2)\\&+\varepsilon\ln\Lambda\Big(\frac{N\beta}{q+1}A_1+\frac{N\alpha}{p+1}A_2\Big)+\varepsilon\Big(\frac{\beta}{2}A_1+\frac{\alpha}{2}A_2+\beta D_1+\alpha D_2\Big)+o(\varepsilon).\end{aligned}$$

Then it remains to show that

$$I_\varepsilon(\Lambda,a)-J_\varepsilon(W_1,W_2)=o(\varepsilon).$$

Actually, a Taylor expansion and the fact that

$$J'_\varepsilon(W_1+\phi_1,W_2+\phi_2)[(\phi_1,\phi_2)]=0,$$

yield

$$\begin{aligned}&I_\varepsilon(\Lambda,a)-J_\varepsilon(W_1,W_2)=J_\varepsilon(W_1+\phi_1,W_2+\phi_2)-J_\varepsilon(W_1,W_2)\\=&-\int_0^1\int_{\Omega_\varepsilon}(|\nabla\phi_1|^2+\mu\varepsilon^2\phi_1^2-(p+\alpha\varepsilon)(W_2+s\phi_2)^{p+\alpha\varepsilon-1}\phi_2^2)sds\\&-\int_0^1\int_{\Omega_\varepsilon}(|\nabla\phi_2|^2+\mu\varepsilon^2\phi_2^2-(q+\beta\varepsilon)(W_1+s\phi_1)^{q+\beta\varepsilon-1}\phi_1^2)sds\\=&-\int_0^1\int_{\Omega_\varepsilon}(N_{\varepsilon,1}(\phi_2)\phi_2+R_{\varepsilon,1}\phi_2+(p+\alpha\varepsilon)[W_2^{p+\alpha\varepsilon-1}-(W_2+s\phi_2)^{p+\alpha\varepsilon-1}]\phi_2^2)sds\\&-\int_0^1\int_{\Omega_\varepsilon}(N_{\varepsilon,2}(\phi_1)\phi_1+R_{\varepsilon,2}\phi_1+(q+\beta\varepsilon)[W_1^{q+\beta\varepsilon-1}-(W_1+s\phi_1)^{q+\beta\varepsilon-1}]\phi_1^2)sds.\end{aligned}$$

The first term can be estimated as follows,

$$\Big|\int_{\Omega_\varepsilon} N_{\varepsilon,1}(\phi_2)\phi_2\Big| \le \begin{cases} C\|\phi_2\|_{*,2}^{p+\alpha\varepsilon+1}\displaystyle\int_{\Omega_\varepsilon}\langle x-\xi\rangle^{-\gamma_2(p+\alpha\varepsilon+1)} \le C\varepsilon^{p+1}, & 1<p\le 2,\\ C\|\phi_2\|_{*,2}^3\displaystyle\int_{\Omega_\varepsilon}\langle x-\xi\rangle^{-3\gamma_2-(p+\alpha\varepsilon-2)(N-2)(1-\tau_2)}+C\varepsilon^{p+1}, & p>2, \end{cases}$$
$$\le C\varepsilon^{\min(3,p+1)},$$

and

$$\Big|\int_{\Omega_\varepsilon} N_{\varepsilon,2}(\phi_1)\phi_1\Big| \le \begin{cases} C\|\phi_1\|_{*,1}^{q+\beta\varepsilon+1}\displaystyle\int_{\Omega_\varepsilon}\langle x-\xi\rangle^{-\gamma_1(q+\beta\varepsilon+1)} \le C\varepsilon^{q+1}, & 1<q\le 2,\\ C\|\phi_1\|_{*,1}^3\displaystyle\int_{\Omega_\varepsilon}\langle x-\xi\rangle^{-3\gamma_1-(q+\beta\varepsilon-2)(N-2)(1-\tau_1)}+C\varepsilon^{q+1}, & q>2, \end{cases}$$
$$\le C\varepsilon^{\min(3,q+1)}.$$

For the second term,

$$R_{\varepsilon,1}\le C\varepsilon\langle x-\xi\rangle^{-(N-3)(1-\tau_2)-(N-2)(p-1)},$$
$$R_{\varepsilon,2}\le C\varepsilon\langle x-\xi\rangle^{-(N-3)(1-\tau_1)-(N-2)(q-1)}.$$

Then

$$\int_{\Omega_\varepsilon}|R_{\varepsilon,1}\phi_2|\le C\varepsilon\|\phi_2\|_{*,2}\int_{\Omega_\varepsilon}\langle x-\xi\rangle^{-\gamma_2-(N-2)p+1}\le C\varepsilon^2,$$
$$\int_{\Omega_\varepsilon}|R_{\varepsilon,2}\phi_1|\le C\varepsilon\|\phi_1\|_{*,1}\int_{\Omega_\varepsilon}\langle x-\xi\rangle^{-\gamma_1-(N-2)q+1}\le C\varepsilon^2.$$

Concerning the last term

$$\begin{aligned}&\int_{\Omega_\varepsilon}[W_2^{p+\alpha\varepsilon-1}-(W_2+s\phi_2)^{p+\alpha\varepsilon-1}]\phi_2^2\\ \le& C\begin{cases}\displaystyle\int_{\Omega_\varepsilon}|\phi_2|^{p+\alpha\varepsilon+1}, & 1<p\le 2,\\ \displaystyle\int_{\Omega_\varepsilon}|\phi_2|^{p+\alpha\varepsilon+1}+W_2^{p+\alpha\varepsilon-2}|\phi_2|^3, & 2<p\le 3,\\ \displaystyle\int_{\Omega_\varepsilon}|\phi_2|^{p+\alpha\varepsilon+1}+W_2^{p+\alpha\varepsilon-2}|\phi_2|^3+W_2^{p+\alpha\varepsilon-3}|\phi_2|^4, & p\ge 3,\end{cases}\\ \le& C\begin{cases}\|\phi_2\|_{*,2}^{p+\alpha\varepsilon+1}\displaystyle\int_{\Omega_\varepsilon}\langle x-\xi\rangle^{-\gamma_2(p+\alpha\varepsilon+1)}\le C\varepsilon^{p+1}, & 1<p\le 2,\\ C\varepsilon^{p+1}+\|\phi_2\|_{*,2}^3\displaystyle\int_{\Omega_\varepsilon}\langle x-\xi\rangle^{-3\gamma_2-(p-2)(N-2)(1-\tau_2)}\le C\varepsilon^3, & 2<p\le 3,\\ C\varepsilon^3+\|\phi_2\|_{*,2}^4\displaystyle\int_{\Omega_\varepsilon}\langle x-\xi\rangle^{-4\gamma_2-(p-3)(N-2)(1-\tau_2)}, & p\ge 3,\end{cases}\\ \le& C\varepsilon^3,\end{aligned}$$

$$
\begin{aligned}
&\int_{\Omega_\varepsilon}[W_1^{q+\beta\varepsilon-1}-(W_1+s\phi_1)^{q+\beta\varepsilon-1}]\phi_1^2\\
\leq& C\begin{cases}
\int_{\Omega_\varepsilon}|\phi_1|^{q+\beta\varepsilon+1}, & 1<q\leq 2,\\
\int_{\Omega_\varepsilon}|\phi_1|^{q+\beta\varepsilon+1}+W_1^{q+\beta\varepsilon-2}|\phi_1|^3, & 2<q\leq 3,\\
\int_{\Omega_\varepsilon}|\phi_1|^{q+\beta\varepsilon+1}+W_1^{q+\beta\varepsilon-2}|\phi_1|^3+W_1^{p+\beta\varepsilon-3}|\phi_1|^4, & q\geq 3,
\end{cases}\\
\leq& C\begin{cases}
\|\phi_1\|_{*,1}^{q+\beta\varepsilon+1}\int_{\Omega_\varepsilon}\langle x-\xi\rangle^{-\gamma_1(q+\beta\varepsilon+1)}\leq C\varepsilon^{q+1}, & 1<q\leq 2,\\
C\varepsilon^{q+1}+\|\phi_1\|_{*,1}^{3}\int_{\Omega_\varepsilon}\langle x-\xi\rangle^{-3\gamma_1-(q-2)(N-2)(1-\tau_1)}\leq C\varepsilon^3, & 2<q\leq 3,\\
C\varepsilon^{3}+\|\phi_1\|_{*,1}^{4}\int_{\Omega_\varepsilon}\langle x-\xi\rangle^{-4\gamma_1-(q-3)(N-2)(1-\tau_1)}, & q\geq 3,
\end{cases}\\
\leq& C\varepsilon^3.
\end{aligned}
$$

The same estimate holds for the first derivative with respect to Λ, then we finish the proof. $\square$

Proof of Theorems 1.1 *and* 1.2. Let $\max_{P\in\partial\Omega}H(P)=\gamma>0$. For $\delta<\gamma$, we define

$$(\partial\Omega)_\delta=\{a\in\partial\Omega\ \text{ s.t. }\ H(a)>\delta\},$$

and

$$
\begin{aligned}
\widehat{I}_\varepsilon(\Lambda,a)=&\Lambda H(a)-\eta\ln\Lambda-\widetilde{\sigma}_\varepsilon(\Lambda,a)\\
=&\frac{1}{\varepsilon(B_1+B_2+C_1+C_2)}\Big(\frac{q-1}{2}A_1+\frac{p-1}{2}A_2-I_\varepsilon(\Lambda,a)\Big)\\
&+\frac{1}{B_1+B_2+C_1+C_2}\Big(\frac{\beta}{2}A_1+\frac{\alpha}{2}A_2+\beta D_1+\alpha D_2\Big),
\end{aligned}
$$

where

$$
\begin{aligned}
&\eta=\frac{1}{B_1+B_2+C_1+C_2}\Big(\frac{N\beta}{q+1}A_1+\frac{N\alpha}{p+1}A_2\Big),\\
&\widetilde{\sigma}_\varepsilon(\Lambda,a)=\frac{\sigma_\varepsilon(\Lambda,a)}{B_1+B_2+C_1+C_2}=o(1),\\
&\partial_\Lambda\widetilde{\sigma}_\varepsilon(\Lambda,a)=o(1),
\end{aligned}
$$

as $\varepsilon\to 0$. We set

$$
\begin{aligned}
&\Sigma_0=\Big\{(\Lambda,a)\Big|\frac{c_1}{2}<\Lambda<\frac{2}{c_1},a\in(\partial\Omega)_{\gamma_0}\Big\}\\
&E=\Big\{(\Lambda,a)\Big|c_1\leq\Lambda\leq\frac{1}{c_1},a\in(\partial\Omega)_{\gamma_1}\Big\}\\
&E_0=\{c_1\}\times(\partial\Omega)_{\gamma_1}\cup\Big\{\frac{1}{c_1}\Big\}\times(\partial\Omega)_{\gamma_1},
\end{aligned}
$$

where c_1 is a small number to be chosen later, $0 < \gamma_0 < \gamma_1 < \gamma$. Here we choose, for γ_1 close enough to γ, a contractible component of $(\partial\Omega)_{\gamma_1}$ so that E is contractible. It is trivial to see that $E_0 \subset E \subset \Sigma_0$, E_0, E are closed and E is connected.

Let Γ be the class of continuous functions $\varphi : E \to \Sigma_0$ with the property that

$$\varphi(y) = y \quad \text{for all } \ y \in E_0.$$

Define the max-min value c as

$$c = \max_{\varphi\in\Gamma} \min_{y\in E} \widehat{I}_\varepsilon(\varphi(y)).$$

We will show that c defines a critical value. We only need to verify the following two conditions

(H1) $\min_{y\in E_0} \widehat{I}_\varepsilon(\varphi(y)) > c, \forall\varphi \in \Gamma$;

(H2) For all $y \in \partial\Sigma_0$ such that $\widehat{I}_\varepsilon(y) = c$, there exists τ_y a tangent vector to $\partial\Sigma_0$ at y such that $\partial_{\tau_y}\widehat{I}_\varepsilon(y) \neq 0$.

Then standard deformation argument ensures that the max-min value c is a topologically nontrivial critical value for $\widehat{I}_\varepsilon(\Lambda, a)$ in Σ_0.

To check (H1) and (H2), we write $\varphi(y) = (\varphi_1(y), \varphi_2(y))$, where $\varphi_1(y) \in [\frac{c_1}{2}, \frac{2}{c_1}]$ and $\varphi_2(y) \in (\partial\Omega)_{\gamma_0}$. Since $\varphi|_{E_0} = id$, E is contractible and φ is continuous, necessarily there is some y in E such that $H(\varphi_2(y)) = \gamma$. Let $(\Lambda_0, a_0) \in E$ such that $H(a_0) = \gamma$, $\Lambda_0 = \frac{\eta}{\gamma}$, where c_1 is chosen small enough so that $\Lambda_0 \in [c_1, \frac{1}{c_1}]$. For any $\varphi \in \Gamma$, φ_1 is a continuous functions from E to $[\frac{c_1}{2}, \frac{2}{c_1}]$ such that $[c_1, \frac{1}{c_1}] \subset \varphi_1(E)$. Thus, there exists $y_0 \in E$ such that $\varphi_1(y_0) = \Lambda_0$.

We note

$$\begin{aligned}\widehat{I}_\varepsilon(\Lambda_0, a_0) = d_0 + o(1) :=& \min\{\widehat{I}_\varepsilon(\Lambda, a), H(a) = \gamma, \Lambda > 0\} + o(1)\\ =&\eta - \eta\ln\eta + \eta\ln\gamma + o(1) \le c.\end{aligned}$$

Moreover,

$$\min_{y\in E} \widehat{I}_\varepsilon(\varphi(y)) \le \widehat{I}_\varepsilon(\Lambda_0, \varphi_2(y)) \le \frac{\eta}{\gamma} H(\varphi_2(y)) - \eta\ln\eta + \eta\ln\gamma + o(1) \le d_0 + o(1).$$

As a consequence

$$c = d_0 + o(1) = \eta - \eta\ln\eta + \eta\ln\gamma + o(1).$$

For $y \in E_0$, we have $\varphi_1(y) = c_1$ or $\varphi_1(y) = \frac{1}{c_1}$. In the first case, we have

$$\widehat{I}_\varepsilon(y) = c_1 H(\varphi_2(y)) - \eta\ln c_1 + o(1) > \eta\ln\frac{1}{c_1} + o(1) > 2d_0 > c,$$

where we choose c_1 is small enough. In the latter case, we have

$$\begin{aligned}\widehat{I}_\varepsilon(y) =&\frac{1}{c_1}H(\varphi_2(y)) + \eta \ln c_1 + o(1)\\ >&\frac{\gamma_1}{c_1} + \eta \ln c_1 + o(1) > 2d_0 > c,\end{aligned}$$

provided c_1 is small enough. So (H1) is verified.

To check (H2), we only need to check conditions on $[c_1, \frac{1}{c_1}] \times \partial((\partial\Omega)_{\gamma_0})$, taking $\tau_y = \frac{\partial}{\partial\Lambda}$, we obtain

$$\partial_{\tau_y}\widehat{I}_\varepsilon(y) = H(y_2) - \frac{\eta}{\Lambda} + o(1) \neq 0.$$

We assume that $\partial_{\tau_y}\widehat{I}_\varepsilon(y) = 0$, then $\Lambda H(y_2) = \eta + o(1)$. On the other hand, we estimate

$$\widehat{I}_\varepsilon(y) = \eta - \eta \ln \eta + \eta \ln H(\varphi_2(y)) + o(1) = \eta - \eta \ln \eta + \eta \ln \gamma_0 + o(1) < c,$$

which is a contradiction to the assumption. So (H2) is also verified.

In case of $\varepsilon < 0$, we assume that Ω is not convex. Similarly as before, we define

$$\begin{aligned}&(\partial\Omega)_\delta = \{a \in \partial\Omega \mid H(a) < -\delta\},\\ &\Sigma_0 = \left\{(\Lambda, a)\middle|\frac{c_1}{2} < \Lambda < \frac{2}{c_1},\ a \in (\partial\Omega)_{\gamma_0}\right\},\\ &E = \left\{(\Lambda, a)\middle|c_1 \le \Lambda \le \frac{1}{c_1},\ a \in (\partial\Omega)_{\gamma_1}\right\},\\ &E_0 = \{c_1\} \times (\partial\Omega)_{\gamma_1} \cup \left\{\frac{1}{c_1}\right\} \times (\partial\Omega)_{\gamma_1},\\ &c = \min_{\varphi\in\Gamma}\max_{y\in E}\widehat{I}_\varepsilon(\varphi(y)),\end{aligned}$$

where $0 < \delta < \gamma = -\min_{a\in\partial\Omega} H(a) > 0$ and $\gamma_0 < \gamma_1 < \gamma$.

(H1) $\max_{y\in E_0} \widehat{I}_\varepsilon(\varphi(y)) < c$, $\forall\varphi \in \Gamma$;

(H2) For all $y \in \partial\Sigma_0$ such that $\widehat{I}_\varepsilon(y) = c$, there exists τ_y a tangent vector to $\partial\Sigma_0$ at y such that $\partial_{\tau_y}\widehat{I}_\varepsilon(y) \neq 0$.

Arguing as previously, we find that c is a critical point of $\widehat{I}_\varepsilon$. This proves Theorem 1.2. □

Appendix

In this paper, we may assume $a = 0 \in \Omega$ fixed, and after rotation and translation of the coordinate system, the inward normal to $\partial\Omega$ at a is the direction of the positive x_N-axis.

Denote $x' = (x_1, \cdots, x_{N-1})$, $B'_\delta(0) = \{x' \in \mathbb{R}^{N-1} : |x'| < \delta\}$ and $B_\delta(a) = \{x \in \mathbb{R}^N : |x-a| < \delta\}$. Then since $\partial\Omega$ is smooth, there exists a constant $\delta > 0$ such that $\partial\Omega \cap B'_\delta(a)$ can be represented by the graph of a smooth function $\rho : B_\delta(a) \to \mathbb{R}$, where $\rho(0) = 0$, $\nabla\rho(0) = 0$, and

$$\Omega \cap B_\delta(a) = \{(x', x_N) \in B_\delta(a) : x_N > \rho(x')\}.$$

Moreover, we set

$$\rho(x') = \frac{1}{2}\sum_{i=1}^{N-1} k_i x_i^2 + \mathcal{O}(|x|^3),$$

where k_i, $i = 1, \cdots, N-1$, are the principal curvatures at a, and then the average of the principal curvatures of $\partial\Omega$ at a is the mean curvature

$$H(a) = \frac{1}{N-1}\sum_{i=1}^{N-1} k_i.$$

On $\partial\Omega \cap B_\delta(a)$, the normal derivative is

$$n(x) = \frac{1}{(1+|\nabla'\rho|^2)^{\frac{1}{2}}}(\nabla'\rho, -1)$$

and the tangential derivatives are given by

$$\frac{\partial}{\partial\tau_{i,x}} = \frac{1}{(1+|\frac{\partial\rho}{\partial x_i}|^2)^{\frac{1}{2}}}\Big(0, \cdots, 1, \cdots, \frac{\partial\rho}{\partial x_i}\Big), \quad i = 1, \cdots, N-1.$$

A Basic estimates

We introduce two auxiliary functions $\varphi_{1,0}$ and $\varphi_{2,0}$:

$$-\Delta\varphi_{1,0} = 0 \quad \text{in } \mathbb{R}^N_+, \quad \frac{\partial\varphi_{1,0}}{\partial x_N} = -\frac{1}{2}\langle\nabla U_{0,1}, kx\rangle|_{x_N=0} \quad \text{on } \partial\mathbb{R}^N_+, \quad \varphi_{1,0} \to 0 \quad \text{as } r \to \infty,$$
$$-\Delta\varphi_{2,0} = 0 \quad \text{in } \mathbb{R}^N_+, \quad \frac{\partial\varphi_{2,0}}{\partial x_N} = -\frac{1}{2}\langle\nabla V_{0,1}, kx\rangle|_{x_N=0} \quad \text{on } \partial\mathbb{R}^N_+, \quad \varphi_{2,0} \to 0 \quad \text{as } r \to \infty,$$

where $kx = (k_1x_1, k_2x_2, \cdots, k_Nx_N)$.

Lemma A.1 ([13]). *For $p > \frac{N}{N-2}$ and $i = 1, 2$, there holds that*

$$|\varphi_{i0}(x)| \le \frac{C}{(1+|x|)^{N-3}}, \quad |\nabla\varphi_{i0}(x)| \le \frac{C}{(1+|x|)^{N-2}}, \quad |D^2\varphi_{i0}(x)| \le \frac{C}{(1+|x|)^{N-1}}.$$

Lemma A.2. *Let $f \in L^t_{\beta+2}(\Omega_\varepsilon)$ and u satisfy*

$$-\Delta u + \mu\varepsilon^2 u = f \quad \textit{in } \Omega_\varepsilon, \quad \frac{\partial u}{\partial n} = 0 \quad \textit{on } \partial\Omega_\varepsilon.$$

Then

$$|u(x)| \leq C \int_\Omega \frac{|f(y)|}{|y-x|^{N-2}} dy, \quad \|u\|_* \leq C\|f\|_{**}.$$

Lemma A.3 ([39, Lemma B.1]). *For any constant* $0 < \sigma \leq \min\{\alpha, \beta\}$*, there is a constant* $C > 0$*, such that*

$$\begin{aligned} &\frac{1}{(1+|y-x_i|)^\alpha} \frac{1}{(1+|y-x_j|)^\beta} \\ \leq &\frac{C}{|x_i - x_j|^\sigma} \Big(\frac{1}{(1+|y-x_i|)^{\alpha+\beta-\sigma}} + \frac{1}{(1+|y-x_j|)^{\alpha+\beta-\sigma}} \Big). \end{aligned}$$

We generalize the estimate used in [39] in the following.

Lemma A.4 (c.f. [12]). *For any constant* $\sigma > 0, \sigma \neq N-2$*, there exists a constant* $C > 0$*, such that*

$$\int_{\mathbb{R}^N} \frac{1}{|y-z|^{N-2}} \frac{1}{(1+|z|)^{2+\sigma}} dz \leq \frac{C}{(1+|y|)^{\min\{\sigma, N-2\}}}.$$

Recall that

$$\begin{cases} -\Delta v_1 + \mu v_1 = V_{a,\lambda}^p & \text{in } \Omega, \\ -\Delta v_2 + \mu v_2 = U_{a,\lambda}^q & \text{in } \Omega, \\ \dfrac{\partial v_1}{\partial n} = \dfrac{\partial v_2}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{A.1}$$

and

$$\begin{cases} v_1 = U_{a,\lambda} - \varphi_1, \\ v_2 = V_{a,\lambda} - \varphi_2. \end{cases} \tag{A.2}$$

Since

$$\begin{cases} -\Delta U_{a,\lambda} = V_{a,\lambda}^p, \\ -\Delta V_{a,\lambda} = U_{a,\lambda}^q, \end{cases} \tag{A.3}$$

then we have

$$\begin{cases} -\Delta \varphi_1 + \mu \varphi_1 = \mu U_{a,\lambda} & \text{in } \Omega, \\ -\Delta \varphi_2 + \mu \varphi_2 = \mu V_{a,\lambda} & \text{in } \Omega, \\ \dfrac{\partial \varphi_1}{\partial n} = \dfrac{\partial U_{a,\lambda}}{\partial n}, \quad \dfrac{\partial \varphi_2}{\partial n} = \dfrac{\partial V_{a,\lambda}}{\partial n} & \text{on } \partial\Omega. \end{cases} \tag{A.4}$$

Lemma A.5. *For* $p > \frac{N}{N-2}$*,* $N \geq 4$*, we have the expansion*

$$\begin{cases} \varphi_1 = (\Lambda\varepsilon)^{1-\frac{N}{q+1}} \varphi_{1,0}\Big(\dfrac{x-a}{\Lambda\varepsilon}\Big) + \mathcal{O}(\varepsilon^{2-\frac{N}{q+1}} |\ln\varepsilon|^m), \\ \varphi_2 = (\Lambda\varepsilon)^{1-\frac{N}{p+1}} \varphi_{2,0}\Big(\dfrac{x-a}{\Lambda\varepsilon}\Big) + \mathcal{O}(\varepsilon^{2-\frac{N}{p+1}} |\ln\varepsilon|^m), \end{cases} \tag{A.5}$$

with

$$m=\begin{cases} 1, & N=4,\\ 0, & N\geq 5.\end{cases} \tag{A.6}$$

Moreover,

$$\begin{cases} |\varphi_1|\leq \dfrac{\varepsilon^{1-\frac{N}{q+1}}|\ln\varepsilon|^n}{(1+|\frac{x-a}{\Lambda\varepsilon}|)^{N-3}},\\ |\varphi_2|\leq \dfrac{\varepsilon^{1-\frac{N}{p+1}}|\ln\varepsilon|^n}{(1+|\frac{x-a}{\Lambda\varepsilon}|)^{N-3}},\end{cases} \tag{A.7}$$

and

$$\begin{cases} |\varphi_1(x)|\leq C(U_{a,\frac{1}{\Lambda\varepsilon}}(x))^{1-\tau},\\ |\varphi_2(x)|\leq C(V_{a,\frac{1}{\Lambda\varepsilon}}(x))^{1-\tau},\end{cases} \tag{A.8}$$

where $n=1$ and $\tau>0$ is any small fixed number for $N=4,5$, $n=0$ and $\tau=0$ for $N\geq 6$.

Proof. We only need to estimate φ_1. Let

$$\varphi_1=\varphi_1^1+\varphi_1^2,$$

where

$$\begin{cases} -\Delta\varphi_1^1+\mu\varphi_1^1=\mu U_{a,\lambda} & \text{in } \Omega,\\ \dfrac{\partial\varphi_1^1}{\partial n}=0 & \text{on } \partial\Omega,\end{cases} \tag{A.9}$$

and

$$\begin{cases} -\Delta\varphi_1^2+\mu\varphi_1^2=0 & \text{in } \Omega,\\ \dfrac{\partial\varphi_1^2}{\partial n}=\dfrac{\partial U_{a,\lambda}}{\partial n} & \text{on } \partial\Omega.\end{cases} \tag{A.10}$$

Set

$$\widehat{\varphi}_1^j(x)=\varepsilon^{\frac{N}{q+1}}\varphi_1^j(\varepsilon x).$$

Then $\widehat{\varphi}_1^1$ satisfies

$$\begin{cases} -\Delta\widehat{\varphi}_1^1+\mu\varepsilon^2\widehat{\varphi}_1^1=\mu\varepsilon^2 U_{\xi,\frac{1}{\Lambda}} & \text{in } \Omega_\varepsilon,\\ \dfrac{\partial\widehat{\varphi}_1^1}{\partial n}=0 & \text{on } \partial\Omega_\varepsilon.\end{cases} \tag{A.11}$$

So

$$\begin{aligned} |\widehat{\varphi}_1^1|\leq& C\varepsilon^2\int_{\Omega_\varepsilon}\frac{U_{\xi,\frac{1}{\Lambda}}}{|x-y|^{N-2}}dy\\ \leq& C\varepsilon^2\int_{\Omega_\varepsilon}\frac{dy}{(1+|y-\xi|)^{N-2}|x-y|^{N-2}}\\ \leq&\frac{C\varepsilon^2|\ln\varepsilon|^{m_1}}{(1+|x-\xi|)^{N-4}},\end{aligned}$$

where

$$m_1=\begin{cases}1, & N=4,\\ 0, & N\geq 5.\end{cases}$$

Consequently,

$$|\varphi_1^1(x)|\leq \frac{C\varepsilon^{2-\frac{N}{q+1}}|\ln\varepsilon|^{m_1}}{(1+|\frac{x-a}{\Lambda\varepsilon}|)^{N-4}}\leq \frac{C\varepsilon^{1-\frac{N}{q+1}}|\ln\varepsilon|^{m_1}}{(1+|\frac{x-a}{\Lambda\varepsilon}|)^{N-3}},$$

and

$$\varphi_1^1=\mathcal{O}(\varepsilon^{2-\frac{N}{q+1}}|\ln\varepsilon|^{m_1}).$$

Next we estimate for φ_1^2. To this end, we write

$$\varphi_1^2=(\Lambda\varepsilon)^{1-\frac{N}{q+1}}\varphi_{1,0}\Big(\frac{x-a}{\Lambda\varepsilon}\Big)+\varphi_1^3+\varphi_1^4,$$

where φ_1^3 satisfies

$$\begin{cases}-\Delta\varphi_1^3+\mu\varphi_1^3=0 & \text{in } \Omega,\\ \dfrac{\partial\varphi_1^3}{\partial n}=\dfrac{\partial U_{a,\lambda}}{\partial n}-\dfrac{\partial}{\partial n}\Big[(\Lambda\varepsilon)^{1-\frac{N}{q+1}}\varphi_{1,0}\Big(\dfrac{x-a}{\Lambda\varepsilon}\Big)\Big] & \text{on } \partial\Omega,\end{cases}\tag{A.12}$$

and φ_1^4 satisfies

$$\begin{cases}-\Delta\varphi_1^4+\mu\varphi_1^4=(\Delta-\mu)\Big[(\Lambda\varepsilon)^{1-\frac{N}{q+1}}\varphi_{1,0}\Big(\dfrac{x-a}{\Lambda\varepsilon}\Big)\Big] & \text{in } \Omega,\\ \dfrac{\partial\varphi_1^3}{\partial n}=0 & \text{on } \partial\Omega.\end{cases}\tag{A.13}$$

The estimate for φ_1^4 is similar to that of φ_1^1,

$$\begin{aligned}|\widehat{\varphi}_1^4|\leq& C\varepsilon^3\Big(\frac{1}{\varepsilon^2}\int_{\Omega_\varepsilon\setminus\mathbb{R}^N_+}\frac{dy}{(1+|y-\xi|)^{N-1}|x-y|^{N-2}}+\int_{\Omega_\varepsilon}\frac{dy}{(1+|y-\xi|)^{N-3}|x-y|^{N-2}}\Big)\\ \leq& C\varepsilon^3\Big(\frac{1}{\varepsilon(1+|x-\xi|)^{N-3}}+\frac{|\ln\varepsilon|^{m_4}}{(1+|x-\xi|)^{N-5}}\Big),\end{aligned}$$

where

$$m_4=\begin{cases}1, & N=5,\\ 0, & N\neq 5.\end{cases}$$

Consequently,

$$|\varphi_1^4(x)|\leq C\Big(\frac{\varepsilon^{2-\frac{N}{q+1}}}{(1+|\frac{x-a}{\Lambda\varepsilon}|)^{N-3}}+\frac{\varepsilon^{3-\frac{N}{q+1}}|\ln\varepsilon|^{m_4}}{(1+|\frac{x-a}{\Lambda\varepsilon}|)^{N-5}}\Big)\leq \frac{C\varepsilon^{1-\frac{N}{q+1}}|\ln\varepsilon|^{m_4}}{(1+|\frac{x-a}{\Lambda\varepsilon}|)^{N-3}},$$

and

$$\varphi_1^4 = \mathcal{O}(\varepsilon^{2-\frac{N}{q+1}}).$$

It remains to estimate φ_1^3.

If $x \in \partial\Omega \cap B_\delta(a)$,

$$\frac{\partial \varphi_1^3}{\partial n} = \mathcal{O}\Big(\frac{\varepsilon^{1-\frac{N}{q+1}}}{(1+\frac{|x'|}{\Lambda\varepsilon})^{N-3}}\Big).$$

If $x \in \partial\Omega \cap B_\delta^c(a)$,

$$\frac{\partial \varphi_1^3}{\partial n} = \mathcal{O}(\varepsilon^{\frac{N}{p+1}}).$$

We use the standard elliptic theory and Green's representation to have

$$|\varphi_1^3| \le C\frac{\varepsilon^{1-\frac{N}{q+1}}}{(1+|\frac{x-a}{\Lambda\varepsilon}|)^{N-3}},$$

and

$$\varphi_1^3 = \mathcal{O}(\varepsilon^{2-\frac{N}{q+1}}).$$

This completes the proof. □

B Energy expansion

Recall that

$$\begin{aligned} J_\varepsilon(W_1,W_2) := &\int_{\Omega_\varepsilon} \nabla W_1 \cdot \nabla W_2 + \mu\varepsilon^2 \int_{\Omega_\varepsilon} W_1 W_2 \\ &- \frac{1}{p+\alpha\varepsilon+1}\int_{\Omega_\varepsilon} |W_2|^{p+\alpha\varepsilon+1} - \frac{1}{q+\beta\varepsilon+1}\int_{\Omega_\varepsilon} |W_1|^{q+\beta\varepsilon+1}. \end{aligned}$$

Proposition B.1. *For $p > \frac{N}{N-2}$, $N \ge 4$, we have the uniform expansions as $\varepsilon \to 0$*

$$\begin{aligned} J_\varepsilon(W_1,W_2) =& \Big(\frac{q-1}{2}A_1 + \frac{p-1}{2}A_2\Big) - \varepsilon\Lambda H(a)(B_1+B_2+C_1+C_2) \\ &+ \varepsilon\ln\Lambda\Big(\frac{N\beta}{q+1}A_1 + \frac{N\alpha}{p+1}A_2\Big) + \varepsilon\Big(\frac{\beta}{2}A_1 + \frac{\alpha}{2}A_2 + \beta D_1 + \alpha D_2\Big) + \mathcal{O}(\varepsilon^{2-\tau}), \\ \frac{\partial J_\varepsilon}{\partial \Lambda} =& \frac{\varepsilon}{\Lambda}\Big(\frac{N\beta}{q+1}A_1 + \frac{N\alpha}{p+1}A_2\Big) - \varepsilon H(a)(B_1+B_2+C_1+C_2) + \mathcal{O}(\varepsilon^{2-\tau}), \end{aligned}$$

where

$$A_1=\frac{1}{q+\beta\varepsilon+1}\int_{\mathbb{R}^N_+}U_{0,1}^{q+1},\qquad A_2=\frac{1}{p+\alpha\varepsilon+1}\int_{\mathbb{R}^N_+}V_{0,1}^{p+1},$$
$$B_1=\frac{q-1}{4(q+\beta\varepsilon+1)}\int_{\partial\mathbb{R}^N_+}U_{0,1}^{q+1}|y|^2,\qquad B_2=\frac{p-1}{4(p+\alpha\varepsilon+1)}\int_{\partial\mathbb{R}^N_+}V_{0,1}^{p+1}|y|^2,$$
$$C_1=-\frac{1}{4}\int_{\partial\mathbb{R}^N_+}\langle\nabla U_{0,1},y\rangle V_{0,1},\qquad C_2=-\frac{1}{4}\int_{\partial\mathbb{R}^N_+}\langle\nabla V_{0,1},y\rangle U_{0,1},$$
$$D_1=-\frac{1}{q+\beta\varepsilon+1}\int_{\mathbb{R}^N_+}U_{0,1}^{q+1}\ln U_{0,1},\qquad D_2=-\frac{1}{p+\alpha\varepsilon+1}\int_{\mathbb{R}^N_+}V_{0,1}^{p+1}\ln V_{0,1}.$$

Proof. For sake of simplicity, we assume that $\varepsilon>0$ and set $U_{\xi,\frac{1}{\Lambda}}=U$, $V_{\xi,\frac{1}{\Lambda}}=V$. We write

$$\begin{aligned}\frac12\int_{\Omega_\varepsilon}\nabla W_1\nabla W_2+\mu\varepsilon^2W_1W_2=&\frac12\int_{\Omega_\varepsilon}(-\Delta W_1+\mu\varepsilon^2W_1)W_2\\=&\frac12\int_{\Omega_\varepsilon}V^pW_2=\frac12\int_{\Omega_\varepsilon}(V^{p+1}-V^p\widehat{\varphi}_2),\end{aligned}$$

and

$$\begin{aligned}\frac12\int_{\Omega_\varepsilon}\nabla W_1\nabla W_2+\mu\varepsilon^2W_1W_2&=\frac12\int_{\Omega_\varepsilon}(-\Delta W_2+\mu\varepsilon^2W_2)W_1\\&=\frac12\int_{\Omega_\varepsilon}U^qW_1=\frac12\int_{\Omega_\varepsilon}(U^{q+1}-U^q\widehat{\varphi}_1).\end{aligned}$$

On the other hand,

$$\begin{aligned}\int_{\Omega_\varepsilon}W_2^{p+\alpha\varepsilon+1}&=\int_{\Omega_\varepsilon}W_2^{p+1}+\int_{\Omega_\varepsilon}W_2^{p+1}(W_2^{\alpha\varepsilon}-1)\\&=\int_{\Omega_\varepsilon}(V-\widehat{\varphi}_2)^{p+1}+\alpha\varepsilon\int_{\Omega_\varepsilon}(V-\widehat{\varphi}_2)^{p+1}\ln(V-\widehat{\varphi}_2)+\mathcal{O}(\varepsilon^{2-\tau_2})\\&=\int_{\Omega_\varepsilon}V^{p+1}-(p+1)\int_{\Omega_\varepsilon}V^p\widehat{\varphi}_2+\alpha\varepsilon\int_{\Omega_\varepsilon}(V-\widehat{\varphi}_2)^{p+1}\ln(V-\widehat{\varphi}_2)+\mathcal{O}(\varepsilon^{2-\tau_2}),\end{aligned}$$

and

$$\begin{aligned}\int_{\Omega_\varepsilon}W_1^{q+\beta\varepsilon+1}&=\int_{\Omega_\varepsilon}W_1^{q+1}+\int_{\Omega_\varepsilon}W_1^{q+1}(W_1^{\beta\varepsilon}-1)\\&=\int_{\Omega_\varepsilon}(U-\widehat{\varphi}_1)^{q+1}+\beta\varepsilon\int_{\Omega_\varepsilon}(U-\widehat{\varphi}_1)^{q+1}\ln(U-\widehat{\varphi}_1)+\mathcal{O}(\varepsilon^{2-\tau_1})\\&=\int_{\Omega_\varepsilon}U^{q+1}-(q+1)\int_{\Omega_\varepsilon}U^q\widehat{\varphi}_1+\beta\varepsilon\int_{\Omega_\varepsilon}(U-\widehat{\varphi}_1)^{q+1}\ln(U-\widehat{\varphi}_1)+\mathcal{O}(\varepsilon^{2-\tau_1}).\end{aligned}$$

Note also that

$$\int_{\Omega_\varepsilon}(V-\widehat{\varphi}_2)^{p+1}\ln(V-\widehat{\varphi}_2)=-\frac{N}{p+1}\ln\Lambda\int_{\mathbb{R}^N_+}V_{0,1}^{p+1}+\int_{\mathbb{R}^N_+}V_{0,1}^{p+1}\ln V_{0,1}+\mathcal{O}(\varepsilon^{1-\tau_2}),$$

and

$$\int_{\Omega_\varepsilon}(U-\widehat{\varphi}_1)^{q+1}\ln(U-\widehat{\varphi}_1)=-\frac{N}{q+1}\ln\Lambda\int_{\mathbb{R}^N_+}U_{0,1}^{q+1}+\int_{\mathbb{R}^N_+}U_{0,1}^{q+1}\ln U_{0,1}+\mathcal{O}(\varepsilon^{1-\tau_1}).$$

Then

$$\begin{aligned}J_\varepsilon(W_1,W_2)=&\Big(\frac{1}{2}-\frac{1}{q+\beta\varepsilon+1}\Big)\int_{\Omega_\varepsilon}U^{q+1}+\Big(\frac{1}{2}-\frac{1}{p+\alpha\varepsilon+1}\Big)\int_{\Omega_\varepsilon}V^{p+1}\\&+\Big(\frac{q+1}{q+\beta\varepsilon+1}-\frac{1}{2}\Big)\int_{\Omega_\varepsilon}U^q\widehat{\varphi}_1+\Big(\frac{p+1}{p+\alpha\varepsilon+1}-\frac{1}{2}\Big)\int_{\Omega_\varepsilon}V^p\widehat{\varphi}_2\\&+\frac{\alpha N}{(p+\alpha\varepsilon+1)(p+1)}\varepsilon\ln\Lambda\int_{\mathbb{R}^N_+}V_{0,1}^{p+1}+\frac{\beta N}{(q+\beta\varepsilon+1)(q+1)}\varepsilon\ln\Lambda\int_{\mathbb{R}^N_+}U_{0,1}^{q+1}\\&-\frac{\alpha}{p+\alpha\varepsilon+1}\varepsilon\int_{\mathbb{R}^N_+}V_{0,1}^{p+1}\ln V_{0,1}-\frac{\beta}{q+\beta\varepsilon+1}\varepsilon\int_{\mathbb{R}^N_+}U_{0,1}^{q+1}\ln U_{0,1}+\mathcal{O}(\varepsilon^{2-\tau}).\end{aligned}$$

We observe that

$$\begin{aligned}\int_{\Omega_\varepsilon}U^{q+1}&=\int_{\mathbb{R}^N_+}U_{0,\frac{1}{\Lambda}}^{q+1}\Big(y',y_N+\frac{\rho(\varepsilon y')}{\varepsilon}\Big)+\mathcal{O}(\varepsilon^{2-\tau_1})\\&=\int_{\mathbb{R}^N_+}U_{0,\frac{1}{\Lambda}}^{q+1}(y',y_N)+\int_{\mathbb{R}^N_+}\frac{\partial U_{0,\frac{1}{\Lambda}}^{q+1}(y',y_N)}{\partial y_N}\frac{\rho(\varepsilon y')}{\varepsilon}+\mathcal{O}(\varepsilon^{2-\tau_1})\\&=\int_{\mathbb{R}^N_+}U_{0,\frac{1}{\Lambda}}^{q+1}-\int_{\partial\mathbb{R}^N_+}U_{0,\frac{1}{\Lambda}}^{q+1}\frac{\rho(\varepsilon y')}{\varepsilon}+O(\varepsilon^{2-\tau_1})\\&=\int_{\mathbb{R}^N_+}U_{0,\frac{1}{\Lambda}}^{q+1}-\int_{\partial\mathbb{R}^N_+}U_{0,\frac{1}{\Lambda}}^{q+1}\Big(\frac{1}{2\varepsilon}\sum_{j=1}^{N-1}k_j(\varepsilon y_j)^2\Big)+\mathcal{O}(\varepsilon^{2-\tau_1})\\&=\int_{\mathbb{R}^N_+}U_{0,\frac{1}{\Lambda}}^{q+1}-\frac{\varepsilon}{2}\int_{\partial\mathbb{R}^N_+}U_{0,\frac{1}{\Lambda}}^{q+1}\Big(\frac{|y|^2}{N-1}\sum_{j=1}^{N-1}k_j\Big)+\mathcal{O}(\varepsilon^{2-\tau_1})\\&=\int_{\mathbb{R}^N_+}U_{0,1}^{q+1}-\frac{\varepsilon\Lambda H(a)}{2}\int_{\partial\mathbb{R}^N_+}U_{0,1}^{q+1}|y|^2+\mathcal{O}(\varepsilon^{2-\tau_1}),\end{aligned}$$

and

$$\int_{\Omega_\varepsilon}V^{p+1}=\int_{\mathbb{R}^N_+}V_{0,1}^{p+1}-\frac{\varepsilon\Lambda H(a)}{2}\int_{\partial\mathbb{R}^N_+}V_{0,1}^{p+1}|y|^2+\mathcal{O}(\varepsilon^{2-\tau_2}).$$

On the other hand,

$$\begin{aligned}\int_{\Omega_\varepsilon}U^q\widehat{\varphi}_1=&\Lambda\varepsilon\int_{\Omega_\varepsilon}U_{0,1}^q\varphi_{1,0}+\mathcal{O}(\varepsilon^{2-\tau_1})\\=&\Lambda\varepsilon\int_{\mathbb{R}^N_+}U_{0,1}^q\varphi_{1,0}+\mathcal{O}(\varepsilon^{2-\tau_1})\\=&\Lambda\varepsilon\int_{\mathbb{R}^N_+}(-\Delta V_{0,1}\varphi_{1,0}+V_{0,1}\Delta\varphi_{1,0})+\mathcal{O}(\varepsilon^{2-\tau_1})\end{aligned}$$

$$
\begin{aligned}
&=\Lambda\varepsilon\int_{\partial\mathbb{R}^N_+}\Big(-\frac{\partial\varphi_{1,0}}{\partial y_N}V_{0,1}\Big)+\mathcal{O}(\varepsilon^{2-\tau_1})\\
&=\frac{\Lambda\varepsilon H(a)}{2}\int_{\partial\mathbb{R}^N_+}\langle\nabla U_{0,1},y\rangle V_{0,1}+\mathcal{O}(\varepsilon^{2-\tau_1}),
\end{aligned}
$$

and

$$
\int_{\Omega_\varepsilon}V^p\widehat{\varphi}_2=\frac{\Lambda\varepsilon H(a)}{2}\int_{\partial\mathbb{R}^N_+}\langle\nabla V_{0,1},y\rangle U_{0,1}+\mathcal{O}(\varepsilon^{2-\tau_2}).
$$

We obtain

$$
\begin{aligned}
J_\varepsilon(W_1,W_2)=&\Big(\frac12-\frac{1}{q+\beta\varepsilon+1}\Big)\Big(\int_{\mathbb{R}^N_+}U_{0,1}^{q+1}-\frac{\varepsilon\Lambda H(a)}{2}\int_{\partial\mathbb{R}^N_+}U_{0,1}^{q+1}|y|^2+\mathcal{O}(\varepsilon^{2-\tau_1})\Big)\\
&+\Big(\frac12-\frac{1}{p+\alpha\varepsilon+1}\Big)\Big(\int_{\mathbb{R}^N_+}V_{0,1}^{p+1}-\frac{\varepsilon\Lambda H(a)}{2}\int_{\partial\mathbb{R}^N_+}V_{0,1}^{p+1}|y|^2+\mathcal{O}(\varepsilon^{2-\tau_2})\Big)\\
&+\Big(\frac12-\frac{\beta\varepsilon}{q+\beta\varepsilon+1}\Big)\Big(\frac{\Lambda\varepsilon H(a)}{2}\int_{\partial\mathbb{R}^N_+}\langle\nabla U_{0,1},y\rangle V_{0,1}+\mathcal{O}(\varepsilon^{2-\tau_1})\Big)\\
&+\Big(\frac12-\frac{\alpha\varepsilon}{p+\alpha\varepsilon+1}\Big)\Big(\frac{\Lambda\varepsilon H(a)}{2}\int_{\partial\mathbb{R}^N_+}\langle\nabla V_{0,1},y\rangle U_{0,1}+\mathcal{O}(\varepsilon^{2-\tau_2})\Big)\\
&+\frac{\alpha N}{(p+\alpha\varepsilon+1)(p+1)}\varepsilon\ln\Lambda\int_{\mathbb{R}^N_+}V_{0,1}^{p+1}+\frac{\beta N}{(q+\beta\varepsilon+1)(q+1)}\varepsilon\ln\Lambda\int_{\mathbb{R}^N_+}U_{0,1}^{q+1}\\
&-\frac{\alpha\varepsilon}{p+\alpha\varepsilon+1}\int_{\mathbb{R}^N_+}V_{0,1}^{p+1}\ln V_{0,1}-\frac{\beta\varepsilon}{q+\beta\varepsilon+1}\int_{\mathbb{R}^N_+}U_{0,1}^{q+1}\ln U_{0,1}+\mathcal{O}(\varepsilon^{2-\tau})\\
=&\frac{q-1+\beta\varepsilon}{2}A_1-\varepsilon\Lambda H(a)B_1+\frac{p-1+\alpha\varepsilon}{2}A_2-\varepsilon\Lambda H(a)B_2\\
&-\varepsilon\Lambda H(a)(C_1+C_2)+\frac{N\ln\Lambda\beta\varepsilon}{q+1}A_1+\frac{N\ln\Lambda\alpha\varepsilon}{p+1}A_2+\beta\varepsilon D_1+\alpha\varepsilon D_2+\mathcal{O}(\varepsilon^{2-\tau})\\
=&\Big(\frac{q-1}{2}A_1+\frac{p-1}{2}A_2\Big)-\varepsilon\Lambda H(a)(B_1+B_2+C_1+C_2)\\
&+\varepsilon\ln\Lambda\Big(\frac{N\beta}{q+1}A_1+\frac{N\alpha}{p+1}A_2\Big)+\varepsilon\Big(\frac{\beta}{2}A_1+\frac{\alpha}{2}A_2+\beta D_1+\alpha D_2\Big)+\mathcal{O}(\varepsilon^{2-\tau}),
\end{aligned}
$$

where

$$
\begin{aligned}
A_1&=\frac{1}{q+\beta\varepsilon+1}\int_{\mathbb{R}^N_+}U_{0,1}^{q+1}, & A_2&=\frac{1}{p+\alpha\varepsilon+1}\int_{\mathbb{R}^N_+}V_{0,1}^{p+1},\\
B_1&=\frac{q-1}{4(q+\beta\varepsilon+1)}\int_{\partial\mathbb{R}^N_+}U_{0,1}^{q+1}|y|^2, & B_2&=\frac{p-1}{4(p+\alpha\varepsilon+1)}\int_{\partial\mathbb{R}^N_+}V_{0,1}^{p+1}|y|^2,\\
C_1&=-\frac14\int_{\partial\mathbb{R}^N_+}\langle\nabla U_{0,1},y\rangle V_{0,1}, & C_2&=-\frac14\int_{\partial\mathbb{R}^N_+}\langle\nabla V_{0,1},y\rangle U_{0,1},\\
D_1&=-\frac{1}{q+\beta\varepsilon+1}\int_{\mathbb{R}^N_+}U_{0,1}^{q+1}\ln U_{0,1}, & D_2&=-\frac{1}{p+\alpha\varepsilon+1}\int_{\mathbb{R}^N_+}V_{0,1}^{p+1}\ln V_{0,1}.
\end{aligned}
$$

Thus we complete the proof. □

Acknowledgements

After our work was received by the journal, we became aware that a paper by A. Pistoia and D. Schiera, identified as arXiv:2407.00794, was posted online on the same day. In their work, part of our results–specifically Theorem 1.1–was independently established using a different approach.

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