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Distributional Boundary Values of Holomorphic Functions on Tubular Domains

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Abstract. The main purpose of this paper is to establish the distributional boundary values of functions in the weighted Hardy space, which improves the results of Carmichael in [4] and [8], where the weight function is linear. As our main result, we will prove that $f(z)$ in $H(\psi,\Gamma)$ has the $\mathcal{Z}'$ boundary value and can be expressed by the inverse Fourier transform of a distribution. Next, we will establish the $\mathcal{S}'$ boundary value under stronger assumptions and give more precise expression if $f(z)$ also converges to $U\in D'_{L^p}(\mathbb{R}^n)$, where $1\le p\le 2$. In addition, we will also study the inverse result, in which we will prove that $f(z)$ is holomorphic on T_Γ.

Key Words: The weighted Hardy space, distributional boundary values, tubular domains.
AMS Subject Classifications: 32A07, 32A40, 42B25, 42B30

1 Introduction

The existence of the distributional boundary values of holomorphic functions on tubular domains plays an important part in the study of complex analysis of several variables. We say a function $f(z)$ holomorphic on the tube $T_\Gamma=\{z\in\mathbb{C}^n : z=x+\mathrm{i}y,\ x\in\mathbb{R}^n,\ y\in\Gamma\}$ has the $\mathcal{D}'$ boundary value $U\in\mathcal{D}'$ if for any compact sub-cone $\Gamma'\subseteq\Gamma$, the limit

$$\lim_{y\in\Gamma',y\to 0}\langle f(x+\mathrm{i}y),\phi(x)\rangle=\langle U,\phi\rangle$$

holds for all $\phi(x)\in\mathcal{D}$, where $\mathcal{D}$ is the space composed of all infinitely differentiable functions on $\mathbb{R}^n$, which have compact support and $\mathcal{D}'$ denotes the space of all linear functionals on $\mathcal{D}$.

Many scholars [13,19] have considered similar problems for different spaces of distributions including $\mathcal{S}'$, the space of tempered distributions [12]. Here, we say an infinitely differentiable function $\varphi(x)$ belongs to $\mathcal{S}$ if

$$\sup_{x\in\mathbb{R}^n}|x^\alpha D^\beta\varphi(x)|<\infty$$

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for any $\alpha,\beta \in \mathbb{N}^n$. Tillmann [20] showed that a function $f(z)$ holomorphic on an octant has the $\mathcal{S}'$ boundary value if $f(z)$ satisfies the boundary condition:

$$|f(z)| \le M\prod_{j=1}^{n}(1+|z_j|^2)^{m_j}|y_j|^{-1/2-k_j},$$

where $M,\ m_j,\ k_j$ are constants.

Beltrami and Wohlers [1–3] obtained the $\mathcal{S}'$ boundary value result for $n=1$ using a boundary condition that is less restrictive than that of Tillmann. More percisely, they proved:

Theorem 1.1. *Suppose $f(z)$ is holomorphic on the upper complex plane $\mathbb{C}^+ = \{z\in\mathbb{C} : z = x+\mathrm{i}y,\ x\in\mathbb{R}^n,\ y>0\}$ and satisfies for any $\delta>0$ that*

$$|f(z)| \le C_\delta(1+|z|)^N \tag{1.1}$$

for all $y\ge\delta$. If $f(z)$ converges in the $\mathcal{S}'$ topology to a generalized function U as $y\to 0^+$, then $U\in\mathcal{S}'$ and U is the inverse Fourier transform of $V\in\mathcal{S}'$ supported in $[0,\infty)$. Moreover, $f(z)=\langle V, \mathrm{e}^{2\pi\mathrm{i}\langle z,t\rangle}\rangle$.

The same result was proved by Dejager [9] in a slightly more general setting. The extension to n dimensions was obtained by Carmichael [6] in the case that if $f(z)$ is holomorphic on the octant $G=\{x+\mathrm{i}y\in\mathbb{C}^n : x\in\mathbb{R}^n,\ y_j>0,\ j=1,2,\cdots,n\}$.

The investigation of the distributional boundary values was also generalized to tubular domains. We now give some definitions that will be used throughout this paper.

A nonempty subset $\Gamma\subseteq\mathbb{R}^n$ is called a cone with vertex at 0 if $\alpha x\in\Gamma$ whenever $x\in\Gamma$ and $\alpha>0$. The dual cone of Γ is expressed as $\Gamma^*=\big\{y\in\mathbb{R}^n : \langle y,x\rangle \ge 0 \text{ for any } x\in\Gamma\big\}$, which is clearly a closed convex cone with vertex at 0. Next, $(\Gamma^*)^*=\overline{\mathrm{ch}(\Gamma)}$, where $\mathrm{ch}(\Gamma)$ is the convex hull of Γ.

We say that the cone Γ is regular if the interior of Γ^* is non-empty. The open cone Γ' is called the compact sub-cone of Γ if $\mathrm{pr}(\overline{\Gamma'})\subset\mathrm{pr}(\Gamma)$, where $\mathrm{pr}(\Gamma)$ is the intersection of Γ and the surface of the unit sphere in $\mathbb{R}^n$.

For any $\beta=(\beta_1,\cdots,\beta_n)\in\mathbb{N}^n$, we denote by D_t^β the differential operation $D_t^\beta = D_{t_1}^{\beta_1}\cdots D_{t_n}^{\beta_n}$, where $D_{t_j}=-\frac{1}{2\pi\mathrm{i}}\frac{\partial}{\partial t_j}$ for $j=1,\cdots,n$.

We say an infinitely differentiable function $\varphi(x)$ belongs to $\mathcal{Z}$ if $\varphi(x)$ can be extended to an entire function, which satisfies for any $\alpha\in\mathbb{N}^n$ that

$$|z^\alpha\varphi(z)| \le M_\beta \exp\{a_1|y_1|+a_2|y_2|+\cdots+a_n|y_n|\},$$

where M_β depends on β and possibly on φ and $a_j>0$ $(j=1,2,\cdots,n)$ depends only on φ.

For any $1\le p<\infty$, a function $\varphi(x)$ infinitely differentiable in $\mathbb{R}^n$ is said to belong to $\mathcal{D}_{L^p}$ if $D^\beta\varphi(x)\in L^p(\mathbb{R}^n)$ for any $\beta\in\mathbb{N}^n$.