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Distributional Boundary Values of Holomorphic Functions on Tubular Domains

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Abstract. The main purpose of this paper is to establish the distributional boundary values of functions in the weighted Hardy space, which improves the results of Carmichael in [4] and [8], where the weight function is linear. As our main result, we will prove that $f(z)$ in $H(\psi,\Gamma)$ has the $\mathcal{Z}'$ boundary value and can be expressed by the inverse Fourier transform of a distribution. Next, we will establish the $\mathcal{S}'$ boundary value under stronger assumptions and give more precise expression if $f(z)$ also converges to $U \in D'_{L^p}(\mathbb{R}^n)$, where $1 \leq p \leq 2$. In addition, we will also study the inverse result, in which we will prove that $f(z)$ is holomorphic on T_Γ.

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1 Introduction

The existence of the distributional boundary values of holomorphic functions on tubular domains plays an important part in the study of complex analysis of several variables. We say a function $f(z)$ holomorphic on the tube $T_\Gamma = \{z \in \mathbb{C}^n : z = x + \mathrm{i}y,\ x \in \mathbb{R}^n,\ y \in \Gamma\}$ has the $\mathcal{D}'$ boundary value $U \in \mathcal{D}'$ if for any compact sub-cone $\Gamma' \subseteq \Gamma$, the limit

$$\lim_{y\in\Gamma', y\to 0} \langle f(x+\mathrm{i}y), \phi(x)\rangle = \langle U, \phi\rangle$$

holds for all $\phi(x) \in \mathcal{D}$, where $\mathcal{D}$ is the space composed of all infinitely differentiable functions on $\mathbb{R}^n$, which have compact support and $\mathcal{D}'$ denotes the space of all linear functionals on $\mathcal{D}$.

Many scholars [13,19] have considered similar problems for different spaces of distributions including $\mathcal{S}'$, the space of tempered distributions [12]. Here, we say an infinitely differentiable function $\varphi(x)$ belongs to $\mathcal{S}$ if

$$\sup_{x\in\mathbb{R}^n} |x^\alpha D^\beta \varphi(x)| < \infty$$

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for any $\alpha,\beta\in\mathbb{N}^n$. Tillmann [20] showed that a function $f(z)$ holomorphic on an octant has the $\mathcal{S}'$ boundary value if $f(z)$ satisfies the boundary condition:

$$|f(z)|\le M\prod_{j=1}^{n}(1+|z_j|^2)^{m_j}|y_j|^{-1/2-k_j},$$

where $M,\ m_j,\ k_j$ are constants.

Beltrami and Wohlers [1–3] obtained the $\mathcal{S}'$ boundary value result for $n=1$ using a boundary condition that is less restrictive than that of Tillmann. More percisely, they proved:

Theorem 1.1. *Suppose $f(z)$ is holomorphic on the upper complex plane $\mathbb{C}^+=\{z\in\mathbb{C}:z=x+\mathrm{i}y,\ x\in\mathbb{R}^n,\ y>0\}$ and satisfies for any $\delta>0$ that*

$$|f(z)|\le C_\delta(1+|z|)^N \tag{1.1}$$

for all $y\ge\delta$. If $f(z)$ converges in the $\mathcal{S}'$ topology to a generalized function U as $y\to 0^+$, then $U\in\mathcal{S}'$ and U is the inverse Fourier transform of $V\in\mathcal{S}'$ supported in $[0,\infty)$. Moreover, $f(z)=\langle V,\mathrm{e}^{2\pi\mathrm{i}\langle z,t\rangle}\rangle$.

The same result was proved by Dejager [9] in a slightly more general setting. The extension to n dimensions was obtained by Carmichael [6] in the case that if $f(z)$ is holomorphic on the octant $G=\{x+\mathrm{i}y\in\mathbb{C}^n:x\in\mathbb{R}^n,\ y_j>0,\ j=1,2,\cdots,n\}$.

The investigation of the distributional boundary values was also generalized to tubular domains. We now give some definitions that will be used throughout this paper.

A nonempty subset $\Gamma\subseteq\mathbb{R}^n$ is called a cone with vertex at 0 if $\alpha x\in\Gamma$ whenever $x\in\Gamma$ and $\alpha>0$. The dual cone of Γ is expressed as $\Gamma^*=\big\{y\in\mathbb{R}^n:\langle y,x\rangle\ge 0\ \text{for any}\ x\in\Gamma\big\}$, which is clearly a closed convex cone with vertex at 0. Next, $(\Gamma^*)^*=\overline{\mathrm{ch}(\Gamma)}$, where $\mathrm{ch}(\Gamma)$ is the convex hull of Γ.

We say that the cone Γ is regular if the interior of Γ^* is non-empty. The open cone Γ' is called the compact sub-cone of Γ if $\mathrm{pr}(\overline{\Gamma'})\subset\mathrm{pr}(\Gamma)$, where $\mathrm{pr}(\Gamma)$ is the intersection of Γ and the surface of the unit sphere in $\mathbb{R}^n$.

For any $\beta=(\beta_1,\cdots,\beta_n)\in\mathbb{N}^n$, we denote by D_t^β the differential operation $D_t^\beta=D_{t_1}^{\beta_1}\cdots D_{t_n}^{\beta_n}$, where $D_{t_j}=-\frac{1}{2\pi\mathrm{i}}\frac{\partial}{\partial t_j}$ for $j=1,\cdots,n$.

We say an infinitely differentiable function $\varphi(x)$ belongs to $\mathcal{Z}$ if $\varphi(x)$ can be extended to an entire function, which satisfies for any $\alpha\in\mathbb{N}^n$ that

$$|z^\alpha\varphi(z)|\le M_\beta\exp\{a_1|y_1|+a_2|y_2|+\cdots+a_n|y_n|\},$$

where M_β depends on β and possibly on φ and $a_j>0$ $(j=1,2,\cdots,n)$ depends only on φ.

For any $1\le p<\infty$, a function $\varphi(x)$ infinitely differentiable in $\mathbb{R}^n$ is said to belong to $\mathcal{D}_{L^p}$ if $D^\beta\varphi(x)\in L^p(\mathbb{R}^n)$ for any $\beta\in\mathbb{N}^n$.

We denote by $\mathcal{Z}'$ and $\mathcal{D}'_{L^q}$ the spaces of all linear functionals on $\mathcal{Z}$ and $\mathcal{D}_{L^p}$ respectively, where $\frac{1}{p}+\frac{1}{q}=1$. Clearly, $\mathcal{Z}\subseteq\mathcal{S}\subseteq\mathcal{D}_{L^p}\subseteq L^p(\mathbb{R}^n)$ and thus $L^q(\mathbb{R}^n)\subseteq\mathcal{D}'_{L^q}\subseteq\mathcal{S}'\subseteq\mathcal{Z}'$.

Carmichael [4, 8] characterized the distributional boundary values of holomorphic functions on tubular domain T_Γ, which satisfies the boundary condition with the weight function $(A+\sigma)|y|$ and achieved an improved version:

Theorem 1.2. *Assume that Γ is an open cone in $\mathbb{R}^n$ and Γ' is any compact sub-cone of Γ. For any $m>0$ and any $\sigma>0$, if $f(z)$ is holomorphic on T_Γ and satisfies the boundary condition:*

$$|f(z)|\leq M(\Gamma',m)(1+|z|)^N\mathrm{e}^{2\pi(A+\sigma)|y|} \tag{1.2}$$

for all $z\in T(\Gamma',m)=\mathbb{R}^n+\mathrm{i}(\Gamma'\backslash\overline{D_n(0,m)})$, where $M(\Gamma',m)$, $A\geq 0$ and N are constants. Then, $f(z)$ has the $\mathcal{Z}'$ boundary value $U\in\mathcal{Z}'$, which is the Fourier transform of $V\in\mathcal{D}'$ having support in $S_A=\{t\in\mathbb{R}^n:\mu_\Gamma(t)\leq A\}$, where $\mu_\Gamma(t)=\inf_{y\in pr(\Gamma)}-\langle t,y\rangle$, and

$$f(z)=\left\langle V,\mathrm{e}^{2\pi\mathrm{i}\langle z,t\rangle}\right\rangle,$$

if $A=0$. Furthermore, the expression is unrestricted by $A\geq 0$ if $f(z)$ converges in $\mathcal{S}'$.

Also, there have been many investigations of the distributional boundary values concerning the $\mathcal{D}'_{L^p}$ topology. For more details see [5–7].

Recently, Qian [17] described distributional boundary values of functions in Hardy space.

Theorem 1.3. *If $f(z)\in H^p(\mathbb{C}^+)$, where $1\leq p\leq\infty$, then, as a tempered distribution, $\widehat{f}$ is supported in $[0,\infty)$.*

In another research [16], Qian et al. proved the converse of Theorem 1.3.

Theorem 1.4. *For $1\leq p\leq\infty$, if $f(x)\in L^p(\mathbb{R})$ and supp $f\subseteq[0,\infty)$ in the distributional sense, then $f(x)$ is the boundary value of a function in $H^p(\mathbb{C}^+)$.*

In [15], Li et al. obtained the equivalent characterization in terms of $H^p(T_\Gamma)$.

Theorem 1.5. *Assume that Γ is a regular open cone in $\mathbb{R}^n$ and $F(x)\in L^p(\mathbb{R}^n)$, where $1\leq p\leq 2$. Then, $F(x)$ is the boundary value of $F(x+\mathrm{i}y)\in H^p(T_\Gamma)$ if and only if supp $\widehat{F}\subseteq\Gamma^*$ in the distributional sense.*

Inspired by Carmichael in [4] and [8], we will extend the boundary condition (1.2) by a measurable function $\psi(y)$, which is defined on an open cone Γ, namely: for any compact sub-cone $\Gamma'\subset\Gamma$ and any $m>0$, there exist $N>0$ and $K(\Gamma',m)$ depending on Γ' and m such that

$$|f(z)|\leq K(\Gamma',m)(1+|z|)^N\mathrm{e}^{2\pi\psi(y)} \tag{1.3}$$

for all $z\in T(\Gamma',m)$.

In our main results, we need to assume that $\psi(y) \in C(\Gamma)$ and satisfies for any compact sub-cone Γ' that

$$R_{\Gamma',\psi} = \limsup_{y\in\Gamma', y\to\infty} \frac{\psi(y)}{|y|} < \infty \tag{1.4}$$

and define a new set $U(\psi,\Gamma)$ by

$$U(\psi,\Gamma) := \Big\{ \xi \in \mathbb{R}^n : \liminf_{y\in\Gamma, y\to\infty} \big(\langle \xi, y\rangle + \psi(y) \big) > -\infty \Big\}.$$

Unless particularly stated, we will also assume that Γ is a regular open convex cone in $\mathbb{R}^n$ from now on. As our first main result in this paper, we will prove:

Theorem 1.6. *Let $\psi(y) \in C(\Gamma)$ such that* (1.4) *holds. If $f(z)$ is holomorphic on T_Γ and satisfies* (1.3)*, then $f(z)$ has the $\mathcal{Z}'$ boundary value $U \in \mathcal{Z}'$, which is the inverse Fourier transform of $V \in \mathcal{D}'$ having support in $\overline{U(\psi,\Gamma)}$. Moreover,*

$$f(z) = \mathfrak{F}^{-1}\big[V e^{-2\pi\langle y,t\rangle}; x \big].$$

Here, the inverse Fourier transform of $W \in \mathcal{D}'$ is defined by

$$\langle \mathfrak{F}^{-1}[W], \varphi \rangle := \langle W, \widehat{\varphi} \rangle$$

for $\varphi \in \mathcal{Z}$, where

$$\widehat{\varphi}(x) = \int_{\mathbb{R}^n} \varphi(t) e^{-2\pi i\langle x,t\rangle} dt.$$

Clearly, the inverse Fourier transform from $\mathcal{D}'$ to $\mathcal{Z}'$ is topologically isomorphic since the Fourier transform from $\mathcal{Z}$ to $\mathcal{D}$ is one-to-one and surjective.

Next, we will strengthen the boundary condition (1.3), which leads us to define the Hardy space $H(\psi,\Gamma)$. We say that a function $f(z)$ holomorphic on T_Γ belongs to $H(\psi,\Gamma)$ if for any compact sub-cone $\Gamma' \subset \Gamma$, there exist $N > 0$ and $K(\Gamma') > 0$ depending only on Γ' such that

$$|f(z)| \le K(\Gamma')(1+|z|)^N e^{2\pi\psi(y)} \tag{1.5}$$

for all $z \in T_{\Gamma'}$.

Theorem 1.7. *Assume that $\psi(y) \in C(\Gamma)$ and satisfies* (1.4)*. If $f(z) \in H(\psi,\Gamma)$, then $f(z)$ has the $\mathcal{Z}'$ boundary value $U \in \mathcal{Z}'$, which is the inverse Fourier transform of $V \in \mathcal{D}'$ having support in $\overline{U(\psi,\Gamma)}$ and*

$$f(z) = \mathfrak{F}^{-1}[V e^{-2\pi\langle y,t\rangle}; x].$$

Restrictions on $\psi(y)$ can only ensure that $f(z)$ has the $\mathcal{Z}'$ boundary value if $f(z)$ is holomorphic on T_Γ and satisfies (1.5). A natural question is that **how to establish stronger assumptions to get the $\mathcal{S}'$ boundary values?** We solve this difficulty by assuming for any compact sub-cone $\Gamma' \subset \Gamma$ that

$$\limsup_{y\in\Gamma', y\to 0} \frac{\psi(y)}{|\log|y||} \le r < \infty. \tag{1.6}$$

Our results are as follows:

Theorem 1.8. *Let $\psi(y) \in C(\Gamma)$ such that* (1.4) *and* (1.6) *holds. If $f(z) \in H(\psi,\Gamma)$, then $f(z)$ has the $\mathcal{S}'$ boundary value $U \in \mathcal{S}'$, which is the inverse Fourier transform of $V \in \mathcal{S}'$ having support in $\overline{U(\psi,\Gamma)}$. Moreover,*

$$f(z) = \langle V, \mathrm{e}^{2\pi \mathrm{i}\langle z,t\rangle}\rangle.$$

Theorem 1.9. *Let $\psi(y) \in C(\Gamma)$ such that* (1.4) *and* (1.6) *holds. If $f(z) \in H(\psi,\Gamma)$ and has the $\mathcal{S}'$ boundary value $U \in D'_{L^p}(\mathbb{R}^n)$, where $1 \le p \le 2$. Then there exists $V = \sum_{|\alpha|\le m} x^\alpha h_\alpha(x) \in \mathcal{S}'$, which is supported in $\overline{U(\psi,\Gamma)}$ such that $U = \mathfrak{F}^{-1}[V]$ and*

$$f(z) = \big\langle V, \mathrm{e}^{2\pi \mathrm{i}\langle z,t\rangle}\big\rangle.$$

Moreover, $h_\alpha(x)$ is bounded and continuous if $p = 1$ and $h_\alpha(x) \in L^q(\mathbb{R}^n)$ if $1 < p \le 2$, where $\frac{1}{p} + \frac{1}{q} = 1$.

Besides, we will also prove that $f(z)$ is holomorphic on T_Γ, which can be regarded as the inverse case of our main theorems.

Theorem 1.10. *Assume that $\psi \in C(\Gamma)$ and satisfies* (1.4)*. Then*

(1). $\mathcal{X}_{U(\psi,\Gamma)}(t)\mathrm{e}^{2\pi \mathrm{i}\langle z,t\rangle} \in L^p(\mathbb{R}^n)$ for any $z \in T_\Gamma$, where $1 \le p \le \infty$;

(2). Let Γ' be any compact sub-cone of Γ and $m \in \mathbb{R}^+$. If $g(t) \in C(\mathbb{R}^n)$, which is supported in $\overline{U(\psi,\Gamma)}$ such that

$$|g(t)| \le M(\Gamma',m)\mathrm{e}^{2\pi\big(\langle \omega,t\rangle + \psi(\omega)\big)} \tag{1.7}$$

for any $\omega \in \Gamma' \backslash \big(\Gamma' \cap \overline{D_n(0,m)}\big)$, where $M(\Gamma',m) > 0$ depends on Γ' and m. Then $\mathrm{e}^{-2\pi\langle y,t\rangle} g(t) \in L^p(\mathbb{R}^n)$ for any $y \in \Gamma$, where $1 \le p \le \infty$.

Theorem 1.11. *Assume that $\psi \in C(\Gamma)$ and satisfies* (1.4)*. For any compact sub-cone $\Gamma' \subset \Gamma$ and any $m > 0$, let $g(t) \in C(\mathbb{R}^n)$, which is supported in $\overline{U(\psi,\Gamma)}$ such that* (1.7) *holds. For given $y_0 \in \Gamma$, define*

$$V := (l(D))^k \big(g(t)\mathrm{e}^{-2\pi\langle y_0,t\rangle}\big),$$

where $l(D)$ denotes derivation polynomial and $k \in \mathbb{N}$. Then $f(z) = \big\langle V, \mathrm{e}^{2\pi \mathrm{i}\langle z,t\rangle}\big\rangle$ is holomorphic on T_Γ and there exist $N_0 > 0$, $\delta = \delta_y > 0$ and $M'(\Gamma',m) > 0$ depending on Γ' and $m > 0$ such that

$$|f(z)| \le M'(\Gamma',m)(1+|z|)^{N_0}\mathrm{e}^{2\pi\psi(\frac{y+y_0}{2})}\mathrm{e}^{\pi R_1(1+\delta)|y|}. \tag{1.8}$$

We organize this paper as follows: In Section 2, we provide some preliminaries. In Section 3, we will establish the $\mathcal{Z}'$ boundary value of $f(z)$ holomorphic on T_Γ and satisfying (1.3) (Theorem 1.6), which will be generalized by Theorem 1.7 using a simpler technique. Then, in Section 4, we will give the $\mathcal{S}'$ boundary value of $f(z) \in H(\psi,\Gamma)$ under stronger assumptions (Theorems 1.8 and 1.9). Finally, Theorems 1.10 and 1.11 will be proved in Section 5 as the inverse case of our main theorems.

2 Preparation of manuscript

In this section, we provide some preliminaries. The following two lemmas are used more than once in our proofs.

Lemma 2.1 ([8]). *Let Γ be an open cone in $\mathbb{R}^n$ and $y \in ch(\Gamma)$. Then there exists $\delta = \delta_y > 0$ depending on y such that for all $t \in \Gamma^*$, there holds*

$$\langle y, t\rangle \geq \delta |y||t|.$$

Furthermore, if Γ' is any compact sub-cone of $ch(\Gamma)$, then there exists $\delta = \delta_{\Gamma'} > 0$ depending only on Γ' such that the above inequality holds for all $y \in \Gamma'$ and all $t \in \Gamma^*$.

By a technique of approximation, we obtain the same result in [11, Lemma 2] if the assumption is reduced to $\psi(y) \in C(\Gamma)$.

Lemma 2.2. *Assume that $\psi(y) \in C(\Gamma)$ and satisfies* (1.4). *Then*

$$U(\psi,\Gamma) \subseteq \Gamma^* + \overline{D_n(0,R_1)},$$

where $D_n(0,R_1)$ denotes the sphere in $\mathbb{R}^n$ centered at 0 *of the radius $R_1 > 0$.*

Proof. Choose $\xi_0 \in U(\xi,\Gamma)$, and we may assume that $\xi_0 \notin \Gamma^*$ (the result is obvious if $\xi_0 \in \Gamma^*$). Then, there exists $\xi_1 \in \Gamma^*$ such that

$$|\xi_1 - \xi_0| = \mathrm{d}(\xi_0, \partial\Gamma^*) = \inf\left\{|\xi_0 - x| : x \in \partial\Gamma^*\right\} > 0$$

and $\langle \xi_1, \xi_0 - \xi_1\rangle = 0$. By geometric projection, we have for all $\widetilde{y} \in \Gamma^*$ that

$$\left\langle \widetilde{y} - \xi_0, \frac{\xi_1 - \xi_0}{|\xi_1 - \xi_0|}\right\rangle \geq |\xi_1 - \xi_0|.$$

Hence

$$\langle \widetilde{y}, \xi_1 - \xi_0\rangle \geq \langle \xi_1, \xi_1 - \xi_0\rangle = 0,$$

which implies that $\xi_1 - \xi_0 \in \overline{\Gamma}$. Next, we choose $\widetilde{\xi_k} \in D_n(\xi_1 - \xi_0, \frac{1}{k}) \cap \Gamma$, where $k > 0$ such that $\frac{1}{k} < \frac{|\xi_1 - \xi_0|}{2}$ and pick $\eta_k = \frac{\widetilde{\xi_k}}{|\widetilde{\xi_k}|} \in \Gamma$. Then, according to (1.4) and the definition of $U(\psi,\Gamma)$, we have for any $\varepsilon > 0$ that there exists $A_{\xi_0} > 0$ and $R_\varepsilon > 0$ such that for all $\rho > R_\varepsilon$, there holds

$$\psi(\rho\eta_k) < \rho(R_1 + \varepsilon)$$

and

$$\langle \xi_0, \rho\eta_k\rangle + \psi(\rho\eta_k) > -A_{\xi_0}.$$

Thus

$$\frac{-A_{\xi_0}}{\rho}\langle \eta_k, \xi_1 - \xi_0\rangle \leq R_1 + \varepsilon + \langle \eta_k, \xi_1\rangle.$$

Finally, taking $\rho \to \infty,\ k \to \infty$ and $\varepsilon \to 0$ orderly, we get that

$$|\xi_1 - \xi_0| \le R_1,$$

which shows $\xi_0 \in \Gamma^* + \overline{D_n(0,R_1)}$. □

The following result is a generalized version in [4] and [8].

Lemma 2.3. *Assume that $\psi(y) \in C(\Gamma)$ and satisfies (1.4). Let $\varphi(x) \in C_c^\infty(\mathbb{R}^n)$, $supp(\varphi) \subset \overline{D_n(0,1)}$, $\varphi(x) \ge 0$ and $\int_{\mathbb{R}^n} \varphi(x)\mathrm{d}x = 1$. For any $0<\varepsilon<1$, write*

$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon^n}\varphi\Big(\frac{x}{\varepsilon}\Big), \quad B_\varepsilon = U(\psi,\Gamma) + \overline{D_n\Big(0,\frac{3\varepsilon}{2}\Big)} \quad and \quad \xi_\varepsilon(t) = X_{B_\varepsilon} * \varphi_{\frac{\varepsilon}{2}}(t),$$

where $X_{B_\varepsilon}(t)$ represents the characteristic function of B_ε. Then $\xi_\varepsilon(t) \in C^\infty(\mathbb{R}^n)$, $\xi_\varepsilon(t) = 1$ in $U(\psi,\Gamma)+\overline{D_n(0,\varepsilon)}$, $\xi_\varepsilon(t)=0$ in the complement of $U(\psi,\Gamma)+\overline{D_n(0,2\varepsilon)}$ and for any $\gamma \in \mathbb{N}^n$ there exists a constant M_γ depending only on γ such that

$$\big|D_t^\gamma \xi_\varepsilon(t)\big| \le M_\gamma. \tag{2.1}$$

Furthermore, $\xi_\varepsilon(t)\mathrm{e}^{2\pi\mathrm{i}\langle z,t\rangle} \in \mathcal{S}$ for all $z \in T^\Gamma$.

Proof. Noting that

$$\xi_\varepsilon(t) = \int_{\overline{D_n(0,\frac{\varepsilon}{2})}} X_{B_\varepsilon}(t-x)\varphi_{\frac{\varepsilon}{2}}(x)\mathrm{d}x,$$

we may assume $x \in \overline{D_n(0,\frac{\varepsilon}{2})}$. If $t \in U(\psi,\Gamma)+\overline{D_n(0,\varepsilon)}$, then $t-x = t_1 + (t_2 - x) \in B_\varepsilon$, where $t_1 \in U(\psi,\Gamma)$ and $t_2 \in \overline{D_n(0,\varepsilon)}$, and thus $\xi_\varepsilon(t)=1$. If $t-x \in B_\varepsilon$, then $t \in U(\psi,\Gamma)+\overline{D_n(0,2\varepsilon)}$ and hence, by contradiction, we get that $\xi_\varepsilon(t)=0$ for $t \notin U(\psi,\Gamma)+\overline{D_n(0,2\varepsilon)}$. Moreover, $\xi_\varepsilon \in C^\infty(\mathbb{R}^n)$ since $X_{B_\varepsilon} \in L^1_{loc}(\mathbb{R}^n)$ and $\varphi_{\frac{\varepsilon}{2}} \in C_c^\infty(\mathbb{R}^n)$. Now, choose $\gamma \in \mathbb{N}^n$, by the fact that $\big|D_x^\gamma \varphi_{\frac{\varepsilon}{2}}(x)\big|$ is bounded, we have

$$|D_t^\gamma \xi_\varepsilon(t)| \le \int_{\overline{D_n(0,\frac{\varepsilon}{2})}} X_{B_\varepsilon}(t-x)\big|D_x^\gamma \varphi_{\frac{\varepsilon}{2}}(x)\big|\mathrm{d}x \le M_\gamma.$$

Next, we will prove $\xi_\varepsilon(t)\mathrm{e}^{2\pi\mathrm{i}\langle z,t\rangle} \in \mathcal{S}$ for any $z \in T^\Gamma$. According to Lemmas 2.1 and 2.2, for any $t \in U(\psi,\Gamma)+\overline{D_n(0,2\varepsilon)}$, there exists $\delta=\delta_y>0$ depending only on y such that

$$\langle t,y\rangle \ge \delta|y||t| - |y|(\delta+1)(R_1+2\varepsilon). \tag{2.2}$$

Hence, for any $\alpha,\ \beta \in \mathbb{N}^n$, there holds

$$\begin{aligned}
&\sup_{t\in U(\psi,\Gamma)+\overline{D_n(0,2\varepsilon)}} \Big|t^\alpha D_t^\beta\big(\xi_\varepsilon(t)\mathrm{e}^{2\pi\mathrm{i}\langle z,t\rangle}\big)\Big| \\
\le &\sum_{\beta_1+\beta_2=\beta}\frac{\beta!}{\beta_1!\beta_2!}\sup_{t\in U(\psi,\Gamma)+\overline{D_n(0,2\varepsilon)}}|z|^{|\beta_2|}|t^\alpha|\big|D_t^{\beta_1}\xi_\varepsilon(t)\big|\mathrm{e}^{-2\pi\langle y,t\rangle} \\
\le &\mathrm{e}^{2\pi(\delta+1)(R_1+2\varepsilon)|y|}\sum_{\beta_1+\beta_2=\beta}\frac{\beta!}{\beta_1!\beta_2!}M_{\beta_1}|z|^{|\beta_2|}\sup_{t\in U(\psi,\Gamma)+\overline{D_n(0,2\varepsilon)}}|t^\alpha|\mathrm{e}^{-2\pi\delta|t||y|}<\infty,
\end{aligned}$$

which completes our proof. □

3 Proof of Theorems 1.6 and 1.7

In this section, we will prove our main result. An essential step in the proof is to set up a well-behaved function $g(t)$ similar to that in [8, Theorem 4.7.1]. In order to deal with it, we will construct a polynomial $l(z)$, which allows us to find $V=(l(D))^k g(t)$ such that $f(z)\to \mathfrak{F}^{-1}[V]$ in $\mathcal{Z}'$ topology.

Proof of Theorem 1.6. Choose a group of basis $e_1,e_2,\cdots,e_n$ in $\mathbb{R}^n$ such that $e_j\in \mathrm{pr}(\Gamma^*)$, $j=1,2,\cdots,n$. For any compact subcone $\Gamma'\subset\Gamma$ and given $y_0\in\mathrm{pr}(\Gamma')$, define

$$l(z):=\langle e_1,z+\mathrm{i}y_0\rangle\langle e_2,z+\mathrm{i}y_0\rangle\cdots\langle e_n,z+\mathrm{i}y_0\rangle,\quad z\in T^{\Gamma'}.$$

Then, by Lemma 2.1, there exists $\sigma=\sigma_{\Gamma'}>0$ such that

$$\begin{aligned}|l(z)|^2=&\big(\langle e_1,x\rangle^2+\langle e_1,y+y_0\rangle^2\big)\cdots\big(\langle e_n,x\rangle^2+\langle e_n,y+y_0\rangle^2\big)\\ \geq&\big(\langle e_1,y\rangle+\sigma\big)^2\cdots\big(\langle e_n,y\rangle+\sigma\big)^2\\ \geq&\sigma^{2n}+\sigma^{2n-2}\langle e_1,y\rangle^2\\ \geq&\sigma^{2n}(1+|y|^2).\end{aligned}$$

Also

$$\begin{aligned}|l(z)|^2\geq&\big(\langle e_1,x\rangle^2+\langle e_1,y_0\rangle^2\big)\cdots\big(\langle e_n,x\rangle^2+\langle e_n,y_0\rangle^2\big)\\ \geq&\big(\langle e_1,x\rangle^2+\sigma^2\big)\cdots\big(\langle e_n,x\rangle^2+\sigma^2\big)\\ \geq&\sigma^{2n}+\sigma^{2n-2}\big(\langle e_1,x\rangle^2+\cdots+\langle e_n,x\rangle^2\big)\\ =&\sigma^{2n}+\sigma^{2n-2}x^TAx,\end{aligned}\tag{3.1}$$

where A is a positive definite matrix. By orthogonal transform, there holds

$$x^TAx\geq\lambda_0|x|^2,$$

where $\lambda_0>0$ is the smallest characteristic root of A. Then (3.1) can be continued as

$$|l(z)|^2\geq m(1+|x|^2),$$

where $m=\min\{\sigma^{2n},\sigma^{2n-2}\lambda_0\}$. As a consequence,

$$|l(z)|^2\geq\frac{m}{4}(1+|z|)^2.$$

Now, choose $k\in\mathbb{N}$ such that $k-N\geq n+1$. In view of $f(z)\in H(\psi,\Gamma)$, we have for any $z\in\mathbb{R}^n+\mathrm{i}\big(\Gamma'\backslash\overline{D_n(0,m)}\big)$ that

$$\begin{aligned}|(l(z))^{-k}f(z)|\leq&K(\Gamma',m)(1+|z|)^N\mathrm{e}^{2\pi\psi(y)}\Big(\frac{m}{4}\Big)^{-\frac{k}{2}}(1+|z|)^{-k}\\ \leq&K'(\Gamma',m)(1+|z|)^{-(n+1)}\mathrm{e}^{2\pi\psi(y)}.\end{aligned}$$

Then, the definition

$$g_y(t) := \int_{\mathbb{R}^n} (l(z))^{-k} f(z) \mathrm{e}^{-2\pi \mathrm{i}\langle z,t\rangle} \mathrm{d}x = \mathrm{e}^{2\pi\langle y,t\rangle} \mathfrak{F}\big[(l(z))^k f(z);t\big]$$

is well done for all $y \in \Gamma' \backslash \overline{D_n(0,m)}$ and thus $g_y(t) \in C_0(\mathbb{R}^n)$. Now, differentiating $g_y(t)$ with respect to y_j $(j=1,2,\cdots,n)$ and using the Cauchy-Riemann equation, we have

$$\begin{aligned}\frac{\partial g_y(t)}{\partial y_j} =& \mathrm{e}^{2\pi\langle y,t\rangle}\left(2\pi t_j \mathfrak{F}\big[(l(z))^k f(z);t\big] + \mathrm{i}\mathfrak{F}\left[\frac{\partial}{\partial x_j}((l(z))^k f(z));t\right]\right)\\ =& \mathrm{e}^{2\pi\langle y,t\rangle}\Big(2\pi t_j \mathfrak{F}\big[(l(z))^k f(z);t\big] + 2\pi \mathrm{i}^2 t_j \mathfrak{F}\big[(l(z))^k f(z);t\big]\Big) = 0,\end{aligned}$$

which shows that $g_y(t)$ doesn't depend on $y \in \Gamma$. For convenience, we denote by $g(t) = g_y(t)$. Next, choose $t_0 \notin \overline{U(\psi,\Gamma)}$, then for given $y_0 \in \mathrm{pr}(\Gamma)$, there exists a sequence $\{\rho_k\} \subset \mathbb{R}^+$ such that

$$\liminf_{\rho_k \to \infty} \langle t_0, \rho_k y_0\rangle + \psi(\rho_k y_0) = -\infty.$$

Thanks to

$$|g(t)| \le K'(\Gamma',m) \mathrm{e}^{2\pi(\langle \rho_k y_0, t_0\rangle + \psi(\rho_k y_0))} \int_{\mathbb{R}^n} \frac{1}{(1+|x|)^{n+1}} \mathrm{d}x,$$

we can immediately obtain that $g(t_0)=0$, i.e., supp $g \subseteq \overline{U(\psi,\Gamma)}$. Also, we have $g(t) \in \mathcal{D}'$. In fact, for $1 \le p \le 2$ and $\frac{1}{p}+\frac{1}{q}=1$, using

$$g(t)\mathrm{e}^{-2\pi\langle y,t\rangle} = \mathfrak{F}\big[(l(z))^{-k} f(z);t\big] \in L^q(\mathbb{R}^n) \cap C_0(\mathbb{R}^n)$$

and $L^q(\mathbb{R}^n) \subseteq \mathcal{S}' \subseteq \mathcal{D}'$, we have $(l(z))^{-k} f(z) \in \mathcal{Z}'$ and that

$$\langle g(t), \phi(t)\rangle := \big\langle \mathrm{e}^{-2\pi<y,t>} g(t), \mathrm{e}^{2\pi\langle y,t\rangle}\phi(t)\big\rangle$$

is well defined. If $\phi_\lambda(t) \to \phi(t)$ in $\mathcal{D}$ as $\lambda \to \lambda_0$, then there exists a compact set $K \subseteq \mathbb{R}^n$ such that supp $\phi_\lambda \subseteq K$, supp $\phi \subseteq K$ and for any $\alpha \in \mathbb{N}^n$, there holds

$$\begin{aligned}&\sup_{t\in K}\Big|D_t^\alpha\big(\mathrm{e}^{2\pi\langle y,t\rangle}\varphi_\lambda(t)\big) - D_t^\alpha\big(\mathrm{e}^{2\pi\langle y,t\rangle}\varphi(t)\big)\Big|\\ \le& \sum_{\alpha_1+\alpha_2=\alpha} \frac{\alpha!}{\alpha_1!\alpha_2!} \sup_{t\in K} \mathrm{e}^{2\pi\langle y,t\rangle} |y|^{|\alpha_1|} \big|D_t^{\alpha_2}(\phi_\lambda(t)-\phi(t))\big| \ \to \ 0,\end{aligned}$$

which shows that $g(t) \in \mathcal{D}'$. Write

$$V := (l(D))^k g(t).$$

Now, we will prove $f(x+\mathrm{i}y)\to\mathfrak{F}^{-1}[V]$ in $\mathcal{Z}'$ as $y\to 0$ in Γ'. Choose $\varphi\in\mathcal{Z}$ and $\phi\in\mathcal{D}$ such that $\varphi=\widehat{\phi}$. Noting $V\in\mathcal{D}'$ and $(l(z))^k\varphi(x)\in\mathcal{Z}$, by exchanging integration and derivation, there holds

$$\begin{aligned}\left\langle\mathfrak{F}^{-1}[V],\varphi\right\rangle&=\left\langle(l(D))^k g(t),\int_{\mathbb{R}^n}\varphi(x)\mathrm{e}^{2\pi\mathrm{i}\langle x,t\rangle}\mathrm{d}x\right\rangle\\&=\left\langle g(t),\int_{\mathbb{R}^n}\varphi(x)(l(x))^k\mathrm{e}^{2\pi\mathrm{i}\langle x,t\rangle}\mathrm{d}x\right\rangle\\&=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}g(t)\varphi(x)(l(x))^k\mathrm{e}^{2\pi\mathrm{i}\langle x,t\rangle}\mathrm{d}t\mathrm{d}x\\&=\left\langle\mathfrak{F}^{-1}[g;x],(l(x))^k\varphi(x)\right\rangle.\end{aligned}$$

Rewrite

$$(l(z))^k=\sum_{|\alpha|\le s}C_\alpha(y)x^\alpha,$$

where s is the degree of $(l(z))^k$ and $C_\alpha(y)$ denotes a polynomial. Then

$$\begin{aligned}(l(z))^k\varphi(x)&=\sum_{|\alpha|\le s}C_\alpha(y)\int_{\mathbb{R}^n}\phi(t)D_t^\alpha\left(\mathrm{e}^{-2\pi\mathrm{i}\langle x,t\rangle}\right)\mathrm{d}x\\&=\sum_{|\alpha|\le s}C_\alpha(y)(-1)^{|\alpha|}\mathfrak{F}[D^\alpha\phi;x].\end{aligned}$$

Noting that $\mathrm{e}^{-2\pi\langle y,t\rangle}D^\alpha\phi\to D^\alpha\phi$ in $\mathcal{D}$ and $C_\alpha(y)-C_\alpha(0)\to 0$ as $y\to 0$ in Γ', we have

$$\begin{aligned}&\left\langle(l(z))^{-k}f(z),\left((l(z))^k-(l(x))^k\right)\varphi(x)\right\rangle\\=&\left\langle\mathfrak{F}^{-1}\left[\mathrm{e}^{-2\pi\langle y,t\rangle}g(t)\right],\sum_{|\alpha|\le s}\left(C_\alpha(y)-C_\alpha(0)\right)(-1)^{|\alpha|}\mathfrak{F}[D^\alpha\phi]\right\rangle\\=&\sum_{|\alpha|\le s}\left(C_\alpha(y)-C_\alpha(0)\right)(-1)^{|\alpha|}\left\langle g(t),\mathrm{e}^{-2\pi\langle y,t\rangle}D^\alpha\phi\right\rangle\quad\to\quad 0.\end{aligned}$$

Hence

$$\begin{aligned}\lim_{y\in\Gamma'\subseteq\Gamma,y\to 0}\left\langle f(x+\mathrm{i}y),\varphi(x)\right\rangle&=\lim_{y\in\Gamma'\subseteq\Gamma,y\to 0}\left\langle(l(z))^{-k}f(z),(l(z))^k\varphi(x)\right\rangle\\&=\lim_{y\in\Gamma'\subseteq\Gamma,y\to 0}\left\langle(l(z))^{-k}f(z),(l(x))^k\varphi(x)\right\rangle\\&=\left\langle\mathfrak{F}^{-1}[g;x],(l(x))^k\varphi(x)\right\rangle\\&=\left\langle\mathfrak{F}^{-1}[V],\varphi\right\rangle.\end{aligned}$$

Finally, by exchanging integration and derivation, we have $f(z)=\mathfrak{F}^{-1}\left[\mathrm{e}^{-2\pi\langle y,t\rangle}V\right]$ in $\mathcal{Z}'$

since

$$
\begin{aligned}
\langle f(z),\varphi(x)\rangle =&\Big\langle (l(z))^k\mathfrak{F}^{-1}[g(t)\mathrm{e}^{-2\pi\langle y,t\rangle};x],\varphi(x)\Big\rangle\\
=&\Big\langle g(t),\mathrm{e}^{-2\pi\langle y,t\rangle}\mathfrak{F}^{-1}[(l(z))^k\varphi(x);t]\Big\rangle\\
=&\Big\langle g(t),\int_{\mathbb{R}^n}(l(-D_t))^k\varphi(x)\mathrm{e}^{-2\pi\langle z,t\rangle}\mathrm{d}x\Big\rangle\\
=&\Big\langle g(t),(l(-D_t))^k\mathfrak{F}^{-1}[\varphi;t]\mathrm{e}^{-2\pi\langle y,t\rangle}\Big\rangle\\
=&\Big\langle (l(D))^k g(t),\mathfrak{F}^{-1}[\varphi;t]\mathrm{e}^{-2\pi\langle y,t\rangle}\Big\rangle\\
=&\Big\langle \mathfrak{F}^{-1}[V\mathrm{e}^{-2\pi\langle y,t\rangle}],\varphi\Big\rangle.
\end{aligned}
$$

The proof is completed. □

Remark 3.1. Take $\psi(y)=A|y|$, where $A\geq 0$, then the boundary condition (1.3) is reduced to (1.2) in Theorem 1.2 and $U(\psi,\Gamma)=\{t\in\mathbb{R}^n:\mu_\Gamma(t)\leq A\}=\Gamma^*+\overline{D_n(0,A)}$. Thus, we can obtain the same result as in Theorem 1.2. Furthermore, let $A=0$ and $n=1$, i.e., $T_\Gamma=\mathbb{C}^+$, then the boundary condition (1.3) is weakened to (1.1) in Theorem 1.1 and $U(\psi,\Gamma)=[0,\infty)$, which shows that Theorem 1.1 is a special case of Theorem 1.6. In addition, if we also let $N=0$, then Theorem 1.3 can be concluded for $p=\infty$.

Our method in Theorem 1.7 is similar but much simpler than Theorem 1.6 for the fact that $g(t)\mathrm{e}^{-2\pi\langle y,t\rangle}\in L^1(\mathbb{R}^n)$, which doesn't hold in Theorem 1.6.

Proof of Theorem 1.7. For any proper sub-cone $\Gamma'\subset\Gamma$ and given $y_0\in\Gamma'$, we can construct a polynomial

$$l(z):=\langle e_1,z+\mathrm{i}y_0\rangle\langle e_2,z+\mathrm{i}y_0\rangle\cdots\langle e_n,z+\mathrm{i}y_0\rangle,$$

and find $k>0$ such that

$$\big|(l(z))^{-k}f(z)\big|\leq K(\Gamma')(1+|z|)^{-(n+1)}\mathrm{e}^{2\pi\psi(y)} \tag{3.2}$$

for all $z\in T_{\Gamma'}$. Again, define

$$g(t):=\int_{\mathbb{R}^n}(l(z))^{-k}f(z)\mathrm{e}^{-2\pi\mathrm{i}\langle z,t\rangle}\mathrm{d}x,$$

then $g(t)\in C(\mathbb{R}^n)$ is supported in $\overline{U(\psi,\Gamma)}$ and doesn't depend on $y\in\Gamma$. Besides, $g(t)\in D'$ and

$$g(t)\mathrm{e}^{-2\pi\langle y,t\rangle}=\mathfrak{F}\big[l(z))^{-k}f(z);t\big]\in L^q(\mathbb{R}^n)\cap C_0(\mathbb{R}^n) \tag{3.3}$$

for $1\leq p\leq 2$, where $\frac{1}{p}+\frac{1}{q}=1$. Set

$$\delta_2:=\inf\big\{\langle x,y\rangle: x\in\mathrm{pr}(\Gamma^*),y\in\overline{D_n(y_0,\delta_1)}\big\}>0,$$

where $\overline{D_n(y_0,\delta_1)} \subseteq \Gamma$, then for all $y \in \overline{D_n(y_0,\delta_1)}$, it follows from Lemma 2.2 and (3.2) that

$$\begin{aligned}\int_{\mathbb{R}^n} \left|g(t)\mathrm{e}^{-2\pi\langle y,t\rangle}\right|\mathrm{d}t \leq & C\int_{\mathbb{R}^n} \mathrm{e}^{2\pi|y'||t|}\mathrm{e}^{-2\pi\langle y,t\rangle}\mathrm{d}t \\ \leq & C\mathrm{e}^{2\pi R_1(\delta_2+|y_0|+\delta_1)}\int_{\mathbb{R}^n} \mathrm{e}^{-2\pi|t|(\delta_2-|y'|)}\mathrm{d}t < \infty,\end{aligned}$$

if $y' \in \Gamma$ such that $0 < |y'| < \delta_2$. In view of the arbitrary of $y_0 \in \Gamma$, we have for all $y \in \Gamma$ that $g(t)\mathrm{e}^{-2\pi\langle y,t\rangle} \in L^1(\mathbb{R}^n)$. Hence, $f(z)$ can be expressed as

$$f(z) = (l(z))^k \int_{\mathbb{R}^n} g(t)\mathrm{e}^{2\pi\mathrm{i}\langle z,t\rangle}\mathrm{d}t$$

for $z \in T_{\Gamma'}$. Now, let

$$V := (l(D))^k g(t).$$

Again, by exchanging integration and derivation, we get for any $\varphi \in \mathcal{Z}$ that

$$\begin{aligned}\big\langle f(z),\varphi(x)\big\rangle =& \Big\langle (l(z))^k \mathfrak{F}^{-1}[g(t)\mathrm{e}^{-2\pi\langle y,t\rangle};x],\varphi(x)\Big\rangle \\ =& \Big\langle g(t),\mathrm{e}^{-2\pi\langle y,t\rangle}\mathfrak{F}^{-1}[(l(z))^k\varphi(x);t]\Big\rangle \\ =& \Big\langle g(t),(l(-D_t))^k\big(\mathrm{e}^{-2\pi\langle y,t\rangle}\mathfrak{F}^{-1}[\varphi(x);t]\big)\Big\rangle \\ =& \Big\langle V,\mathrm{e}^{-2\pi\langle y,t\rangle}\mathfrak{F}^{-1}[\varphi(x);t]\Big\rangle \\ \to & \Big\langle V,\mathfrak{F}^{-1}[\varphi(x);t]\Big\rangle \\ =& \Big\langle \mathfrak{F}^{-1}[V],\varphi(x)\Big\rangle\end{aligned}$$

as $y \to 0$ in Γ'. Finally, the proof of $f(z) = \mathfrak{F}^{-1}[V\mathrm{e}^{-2\pi\langle y,t\rangle};x]$ in $\mathcal{Z}'$ is similar to Theorem 1.7. $\square$

4 Proof of Theorems 1.8 and 1.9

In this section, we will investigate the $\mathcal{S}'$ boundary value of analytic function $f(z)$, which satisfies (1.5) as well as the precise expression if $f(z)$ also converges in $\mathcal{S}'$ to $U \in D'_{L^p}$.

Proof of Theorem 1.8. We may construct a polynomial $l(z)$ and define $g(t)$ similar to Theorem 1.7. From (1.6), there exists $0 < \delta < 1$ such that for all $|t| > \frac{r}{\delta}$, there holds

$$\inf_{y\in\Gamma',|y|<\delta}\big(\langle y,t\rangle + \psi(y)\big) \leq \inf_{y\in\Gamma',|y|<\delta}\big(|y||t| - r\log|y|\big) = r + r\log\frac{|t|}{r}.$$

Hence, by (3.2), we have

$$|g(t)| \leq K'(\Gamma')\mathrm{e}^{2\pi\inf\{\langle y,t\rangle+\psi(y):y\in\Gamma',|y|<\delta\}} \leq C_0(1+|t|)^{2\pi r}.$$

If $|t| \le \frac{r}{\delta}$, we also have $|g(t)| \le C_0'(1+|t|)^{2\pi r}$ since $g(t) \in C(\mathbb{R}^n)$. As a consequence, $g(t)$ is a tempered L^p function and thus $g(t) \in \mathcal{S}'$. For any $\varphi \in \mathcal{S}$, we have

$$\xi_\varepsilon(t)\mathrm{e}^{-2\pi\langle y,t\rangle}\varphi(t) \;\;\to\;\; \xi_\varepsilon(t)\varphi(t)$$

in $\mathcal{S}'$ as $y \to 0$ in Γ', where $\xi_\varepsilon(t)$ is that in Lemma 2.3. In fact, using Lemmas 2.1 and 2.2, we have for any $\alpha,\ \beta \in \mathbb{N}^n$ that

$$\begin{aligned}
&\sup_{t\in\mathbb{R}^n} |t^\alpha|\big|D_t^\beta(\xi_\varepsilon(t)\mathrm{e}^{-2\pi\langle y,t\rangle}\varphi(t)-\xi_\varepsilon(t)\varphi(t))\big| \\
\le&\sum_{\beta_1+\beta_2=\beta} C_{\beta_1,\beta_2}\sup_{t\in E}|t^\alpha|\big|D_t^{\beta_1}\xi_\varepsilon(t)\big|\big|D_t^{\beta_2}\varphi(t)\big|\big|\mathrm{e}^{-2\pi\langle y,t\rangle}-1\big| \\
&\quad+\sum_{\beta_1+\beta_2+\beta_3=\beta} C_{\beta_1,\beta_2,\beta_3}\sup_{t\in E}|t^\alpha|\big|D_t^{\beta_1}\xi_\varepsilon(t)\big|\big|D_t^{\beta_2}\varphi(t)\big||y^{\beta_3}|\mathrm{e}^{-2\pi\langle y,t\rangle} \\
\le&\sum_{\beta_1+\beta_2=\beta} C'_{\beta_1,\beta_2}\sup_{t\in E}|t^\alpha|\big|D_t^{\beta_2}\varphi(t)\big|\big|\mathrm{e}^{-2\pi\langle y,t\rangle}-1\big| \\
&\quad+\sum_{\beta_1+\beta_2+\beta_3=\beta} C'_{\beta_1,\beta_2,\beta_3}|y|^{|\beta_3|}\mathrm{e}^{2\pi(R_1+2\varepsilon)|y|(\delta_{\Gamma'}+1)}\sup_{t\in E}|t^\alpha|\mathrm{e}^{-2\pi\delta_{\Gamma'}|t||y|} \;\;\to\;\; 0
\end{aligned}$$

as $y \to 0$ in Γ', where $E = U(\psi,\Gamma) + \overline{D_n(0,2\varepsilon)}$. Thus

$$\begin{aligned}
\big\langle f(x+\mathrm{i}y),\varphi(x)\big\rangle =&\big\langle (l(z))^k\mathfrak{F}^{-1}[g(t)\mathrm{e}^{-2\pi\langle y,t\rangle};x],\varphi(x)\big\rangle \\
=&\big\langle g(t),\xi_\varepsilon(t)\mathrm{e}^{-2\pi\langle y,t\rangle}\mathfrak{F}^{-1}[(l(z))^k\varphi(x);t]\big\rangle \\
=&\Big\langle g(t),\xi_\varepsilon(t)\int_{\mathbb{R}^n}\varphi(x)(l(-D_t))^k\mathrm{e}^{2\pi\mathrm{i}\langle z,t\rangle}\mathrm{d}x\Big\rangle.
\end{aligned}$$

Noting that

$$|\varphi(x)(l(z))^k\mathrm{e}^{2\pi\mathrm{i}\langle z,t\rangle}| \le |\varphi(x)(l(z))^k|\mathrm{e}^{(R_1+2\varepsilon)|y|(\delta_0+1)} \in L^1(\mathbb{R}^n),$$

we can continue the above equality as

$$\begin{aligned}
\big\langle f(x+\mathrm{i}y),\varphi(x)\big\rangle =&\Big\langle g(t),\xi_\varepsilon(t)(l(-D_t))^k\int_{\mathbb{R}^n}\varphi(x)\mathrm{e}^{2\pi\mathrm{i}\langle z,t\rangle}\mathrm{d}x\Big\rangle \\
=&\big\langle (l(D))^k g(t),\xi_\varepsilon(t)\mathrm{e}^{-2\pi\langle y,t\rangle}\mathfrak{F}^{-1}[\varphi;t]\big\rangle \\
\to&\;\big\langle (l(D))^k g(t),\xi_\varepsilon(t)\mathfrak{F}^{-1}[\varphi;t]\big\rangle \\
=&\big\langle \mathfrak{F}^{-1}[V],\varphi\big\rangle.
\end{aligned} \tag{4.1}$$

Next, by similar approach in Theorem 1.7, we can prove that $f(z) = \mathfrak{F}^{-1}[V\mathrm{e}^{-2\pi\langle y,t\rangle};x]$ in

$\mathcal{S}'$. Finally, we also have

$$\begin{aligned}\left\langle V, \mathrm{e}^{2\pi \mathrm{i}\langle z,t\rangle}\right\rangle =&\left\langle (l(D))^k g(t), \xi_\varepsilon(t)\mathrm{e}^{2\pi \mathrm{i}\langle z,t\rangle}\right\rangle\\ =&\left\langle g(t), (l(-D_t))^k\left(\xi_\varepsilon(t)\mathrm{e}^{2\pi \mathrm{i}\langle z,t\rangle}\right)\right\rangle\\ =&\left\langle g(t), \xi_\varepsilon(t)(l(z))^k\mathrm{e}^{2\pi \mathrm{i}\langle z,t\rangle}\right\rangle\\ =&(l(z))^k\int_{\mathbb{R}^n} g(t)\mathrm{e}^{2\pi \mathrm{i}\langle z,t\rangle}\mathrm{d}t = f(z),\end{aligned}$$

which completes our proof. □

Remark 4.1. The Montel space [14, 18] is the topological vector space, which is locally convex and separable. We say $\mathcal{B} \subseteq \mathcal{S}$ is bounded if for any $\alpha,\ \beta \in \mathbb{N}^n$ and any $\varphi \in \mathcal{S}$, there holds

$$\sup_{\varphi\in\mathcal{B},t\in\mathbb{R}^n}\left|t^\alpha D_t^\beta\varphi(t)\right| < \infty.$$

Since $\mathcal{S}'$ is a Montel space [18], we can conclude that (4.1) holds uniformly on any bounded set $\mathcal{B}$ of $\mathcal{S}$.

Proof of Theorem 1.9. Using the characterization theorem of Schwartz [18], there exists an integer $m > 0$ such that

$$U = \sum_{|\alpha|\le m} D_t^\alpha g_\alpha(t),$$

where $g_\alpha(t) \in L^p(\mathbb{R}^n)$. According to Theorem 1.8, there exists $V \in \mathcal{S}'$, which is supported in $\overline{U(\psi,\Gamma)}$ such that $U = \mathfrak{F}^{-1}[V]$ and

$$f(z) = \langle V, \mathrm{e}^{2\pi \mathrm{i}\langle z,t\rangle}\rangle.$$

Also, we have for any $\phi(x) \in \mathcal{S}$ that

$$\begin{aligned}\langle V, \phi(x)\rangle =&\langle U, \mathfrak{F}[\phi(x);t]\rangle\\ =&\sum_{|\alpha|\le m}\int_{\mathbb{R}^n} D_t^\alpha g_\alpha(t)\mathfrak{F}[\phi(x);t]\mathrm{d}t\\ =&\sum_{|\alpha|\le m}\int_{\mathbb{R}^n} g_\alpha(t)\mathfrak{F}[x^\alpha\phi(x);t]\mathrm{d}t\\ =&\sum_{|\alpha|\le m}\langle \mathfrak{F}[g_\alpha(t);x], x^\alpha\phi(x)\rangle\\ =&\left\langle \sum_{|\alpha|\le m} x^\alpha h_\alpha(x), \phi(x)\right\rangle.\end{aligned}$$

Therefore, $V = \sum_{|\alpha|\le m} x^\alpha h_\alpha(x) \in \mathcal{S}'$, where $h_\alpha(x) = \mathfrak{F}[g_\alpha(t);x]$. Finally, we conclude $h_\alpha(x) \in L^q(\mathbb{R}^n)$ for $1 < p \le 2$ and that $h_\alpha(x)$ is a bounded continuous function for $p = 1$. □

5 Proof of Theorems 1.10 and 1.11

Proof of Theorem 1.10. Part (1) can be easily achieved by (2.2) since

$$\left|\mathcal{X}_{U(\psi,\Gamma)}(t)\mathrm{e}^{2\pi\mathrm{i}\langle z,t\rangle}\right| \le \mathcal{X}_{U(\psi,\Gamma)}\mathrm{e}^{-2\pi\delta|y||t|}\mathrm{e}^{2\pi|y|(\delta+1)R_1} \le \mathrm{e}^{2\pi|y|(\delta+1)R_1}.$$

Proof of Part (2). For any $y\in\Gamma$, there exist a compact subcone $\Gamma'\subset\Gamma$ and $m>0$ such that $y\in\Gamma'\backslash\big(\Gamma'\cap\overline{D_n(0,m)}\big)$. Choose $\lambda>0$ such that $\frac{m}{|y|}<\lambda<1$ and pick $\omega=\lambda y$, we get

$$|g(t)|\le M(\Gamma',m)\mathrm{e}^{2\pi\left(\lambda\langle y,t\rangle+\psi(\lambda y)\right)}.$$

Again, by (2.2), there holds

$$\begin{aligned}|\mathrm{e}^{-2\pi\langle y,t\rangle}g(t)| \le& M(\Gamma',m)\mathrm{e}^{2\pi\psi(\lambda y)}\mathrm{e}^{2\pi(1-\lambda)R_1|y|(\delta+1)}\mathrm{e}^{-2\pi(1-\lambda)\delta|y||t|}\\ \le& M(\Gamma',m)\mathrm{e}^{2\pi\psi(\lambda y)}\mathrm{e}^{2\pi(1-\lambda)R_1|y|(\delta+1)},\end{aligned}$$

which implies that $\mathrm{e}^{-2\pi\langle y,t\rangle}g(t)\in L^p(\mathbb{R}^n)$ for all $1\le p\le\infty$. □

Proof of Theorem 1.11. By Lemma 2.3, Theorem 1.10 and supp $g\subseteq\overline{U(\psi,\Gamma)}$, we have for all $z\in T^\Gamma$ that

$$\begin{aligned}\left\langle g(t)\mathrm{e}^{-2\pi\langle y_0,t\rangle},\mathrm{e}^{2\pi\langle z,t\rangle}\right\rangle =&\left\langle g(t)\mathrm{e}^{-2\pi\langle y_0,t\rangle},\xi_\varepsilon(t)\mathrm{e}^{2\pi\langle z,t\rangle}\right\rangle\\ =&\int_{U(\psi,\Gamma)}g(t)\mathrm{e}^{-2\pi\langle y_0,t\rangle}\mathrm{e}^{2\pi\langle z,t\rangle}\mathrm{d}t.\end{aligned}$$

Thus

$$\begin{aligned}&(l(z))^k\int_{U(\psi,\Gamma)}g(t)\mathrm{e}^{-2\pi\langle y_0,t\rangle}\mathrm{e}^{2\pi\mathrm{i}\langle z,t\rangle}\mathrm{d}t\\ =&\int_{\mathbb{R}^n}g(t)\mathrm{e}^{-2\pi\langle y_0,t\rangle}(l(-D))^k\mathrm{e}^{2\pi\mathrm{i}\langle z,t\rangle}\mathrm{d}t\\ =&\int_{\mathbb{R}^n}(l(D))^k\big(g(t)\mathrm{e}^{-2\pi\langle y_0,t\rangle}\big)\mathrm{e}^{2\pi\mathrm{i}\langle z,t\rangle}\mathrm{d}t=f(z).\end{aligned}$$

For given $z'=x'+\mathrm{i}y'\in T^\Gamma$, we can find a compact subcone $\Gamma'\subset\Gamma$, $\delta_1>0$, $m>0$, $b>0$ and $d>0$ such that $\overline{D_n(y',\delta_1)}\subset\Gamma'\backslash\big(\Gamma'\cap\overline{D_n(0,m)}\big)\subset\Gamma$ and

$$y+y_0\in\Gamma',\quad 0<m<b<|y+y_0|<d,$$

for all $y\in\overline{D_n(y',\delta_1)}$. Now, we pick $\omega_0=\lambda(y+y_0)$, where $\lambda=\frac{m}{b}$, then $\omega_0\in\Gamma'\backslash\big(\Gamma'\cap\overline{D_n(0,m)}\big)$ and $|\omega_0|<d$. Using (1.7), Lemmas 2.1 and 2.2, there holds

$$\begin{aligned}&\left|g(t)\mathrm{e}^{-2\pi\langle y_0,t\rangle}\mathrm{e}^{2\pi\mathrm{i}\langle z,t\rangle}\right|\\ \le& M(\Gamma',m)\mathrm{e}^{2\pi\psi(\omega_0)}\mathrm{e}^{-2\pi(1-\lambda)\langle y+y_0,t\rangle}\\ \le& M(\Gamma',m)\mathrm{e}^{2\pi\psi(\omega_0)+2\pi(1-\lambda)R_1d(1+\delta)}\mathrm{e}^{-2\pi(1-\lambda)\delta b|t|},\end{aligned}$$

which implies that $g(t)\mathrm{e}^{-2\pi\langle y_0,t\rangle}\mathrm{e}^{2\pi\mathrm{i}\langle z,t\rangle} \in L^1(\mathbb{R}^n)$ since $\psi(\omega_0)=\psi(\lambda(y+y_0))$ is bounded in $\overline{D_n(y',\delta_1)}$. Hence, by Lebesgue's dominated convergence theorem, we have

$$\int_{U(\psi,\Gamma)} g(t)\big(\mathrm{e}^{2\pi\mathrm{i}\langle z,t\rangle}-\mathrm{e}^{2\pi\mathrm{i}\langle z',t\rangle}\big)\mathrm{d}t \ \rightarrow \ 0$$

as $z \rightarrow z'$ in $T^{\overline{D_n(y',\delta_1)}}$ and thus $f(z)$ is continuous on T^Γ. Now, applying Morera theorem [10] with respect to each variable z_j $(j=1,2,\cdots,n)$, we conclude that $f(z)$ is analytic in T^Γ.

Similarly, if we replace m by $\frac{m}{2}$ and set $\omega=\frac{y+y_0}{2}\in\Gamma'\backslash\big(\Gamma'\cap\overline{D_n(0,m)}\big)$, then (1.8) holds since

$$\begin{aligned}|f(z)|=&\left|(l(z))^k\int_{U(\psi,\Gamma)} g(t)\mathrm{e}^{-2\pi\langle y_0,t\rangle}\mathrm{e}^{2\pi\mathrm{i}\langle z,t\rangle}\mathrm{d}t\right|\\ \leq&|(l(z))|^k M\Big(\Gamma',\frac{m}{2}\Big)\mathrm{e}^{2\pi\psi\left(\frac{y+y_0}{2}\right)}\int_{U(\psi,\Gamma)}\mathrm{e}^{-\pi\langle y+y_0,t\rangle}\mathrm{d}t\\ \leq&M'(\Gamma',m)(1+|z|)^N\mathrm{e}^{2\pi\psi(\frac{y+y_0}{2})}\mathrm{e}^{\pi R_1(\delta+1)|y+y_0|},\end{aligned}$$

where N_0 denotes the degree of $(l(z))^k$. □

Remark 5.1. If $0<s<1$, then $\mathrm{e}^{2\pi\psi(\frac{y+y_0}{2})}$ in (1.8) can be generalized by $\mathrm{e}^{2\pi\psi(sy+sy_0)}$.

Remark 5.2. Take $n=1$, i.e., $T_\Gamma=[0,\infty)$, $\psi(y)=0$, $y_0=0$ and $k=0$, then $g(t)\in C(\mathbb{R})$ is supported in $[0,\infty)$ and (1.7) indicates that $g(t)\in L^\infty(\mathbb{R})$. The expression $f(z)=\langle V,\mathrm{e}^{2\pi\mathrm{i}\langle z,t\rangle}\rangle$ is reduced to

$$f(z)=\mathfrak{F}^{-1}[g(t)\mathrm{e}^{-2\pi yt};x]$$

and thus Theorem 1.11 surprisingly improves the result in Theorem 1.4 for $p=\infty$.

References

[1] E. J. Beltrami and M. R. Wohlers, Distributional boundary value of holomorphic in a half plane, J. Math. Mech., (1966), 137–146.

[2] E. J. Beltrami and M. R. Wohlers, Distributional boundary value theorems and Hillert transforms, Arch. Ration. Mech. Anal., 4 (1965), 304–309.

[3] E. J. Beltrami, M. R. Wohlers and G. Sposito, Distributions and the Boundray Values of Analytic Functions, Academic Press, 1966.

[4] R. D. Carmichael, Distibutional boundary values of functions analytic in tubular radial domains, Indiana Univ. Math. J., 9 (1976), 843–853.

[5] R. D. Carmichael, Distibutional boundary values in $\mathcal{D}'_{L^p}$, Rend. Semin. Mat. Univ. Padova, (1970), 35–53.

[6] R. D. Carmichael, Functions analytic in an octant and boundary values of distributions, J. Math. Analy. Appl., 3 (1971), 616–626.

[7] R. D. Carmichael, Generalized Cauchy and Pission integrals and sidtributional boundary values, SIAM J. Math. Anal., 1 (1971), 198–219.

[8] R. D. Carmichael and D. Mitrović, Distributions and analytic functions, Bull. Am. Math. Soc., (1991), 162–170.
[9] E. M. Dejager, The Lorentz-Invariant solutions to the Klein-Gordon equation, SIAM J. Appl. Math., 4 (1967), 944–963.
[10] G. T. Deng, Complex Analysis, Beijing Normal Universty Press, 2010 (in Chinese).
[11] G. T. Deng, Y. Huang and T. Qian, Paley-Wiener-type theorem for analytic functions in tubular domains, J. Math. Anal. Appl., (2019), 123367.
[12] Y. Ding, Foundation of Moderm Analytic, Beijing Normal Universty Press, 2013 (in Chinese).
[13] H. A. Lauwerier, The Hilbert problem for generalized functions, Arch. Ration. Mech. Anal., 1 (1963), 157–166.
[14] R. E. Edwards, Functional analysis, theory and applications, Amer. Math. Mon., 1 (1968), 92.
[15] H. C. Li, G. T. Deng, and T. Qian, Fourier spectrum characterizations of H^p space on tubes over cones for $1 \leq p \leq \infty$, Complex Anal. Oper. Theory, (2018), 1193–1218.
[16] T. Qian, Y. S. Xu, D. Y. Yan, L. X. Yan, and B. Yu, Fourier spectrum characterization of Hardy spaces and applications, Trans. Amer. Math. Soc., 3 (2009), 971–980.
[17] T. Qian, Characterization of boundary values of functions in Hardy spaces with applications in signal analysis, J. Integral Equations Appl., 5 (2015), 159–198.
[18] L. Schwartz, Théorie Des Distributions, Hermann, Paris, 1966.
[19] E. M. Stein and G. Weiss, Introduction to Fourier Analysis on Euclidean Spaces, Princeton University Press, 1971.
[20] H. G. Tillmann, Darstellung der Schwartzschen Distributionen durch analytische Funktionen, Math. Z., 1 (1961), 106–124.