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On Solutions of Differential-Difference Equations in $\mathbb{C}^n$

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Abstract. In this paper, we mainly explore the existence of entire solutions of the quadratic trinomial partial differential-difference equation

$$af^2(z)+2\omega f(z)(a_0f(z)+L_{1,2}^{k+s}(f(z)))+b(a_0f(z)+L_{1,2}^{k+s}(f(z)))^2=e^{g(z)}$$

by utilizing Nevanlinna's theory in several complex variables, where $g(z)$ is entire functions in $\mathbb{C}^n$, $\omega\neq 0$ and $a,b,\omega\in\mathbb{C}$. Furthermore, we get the exact froms of solutions of the above differential-difference equation when $\omega=0$. Our results are generalizations of previous results. In addition, some examples are given to illustrate the accuracy of the results.

Key Words: Differential-difference equations, Nevanlinna theory, finite order, entire solutions.

AMS Subject Classifications: 39A45, 30D35, 39A14, 32H30, 35A20

1 Introduction and main results

In this paper, f denotes a meromorphic function in $\mathbb{C}^n$. We assume that the reader is already familiar with the relevent symbols and concepts of Nevanlinna's value distribution theory [1,2,7], such as the proximate function $m(r,f)$, the counting function $N(r,f)$, the reduced counting function $\overline{N}(r,f)$, the characteristic function $T(r,f)$ in $\mathbb{C}^n$, and $S(r,f)$ denotes the quantity satisfying $S(r,f)=o(T(r,f))$ as $r\to\infty$ outside of a possible exceptional set of finite linear measure.

We denote $z+c=(z_1+c_1,\cdots,z_n+c_n)$ for any $z=(z_1,\cdots,z_n)\in\mathbb{C}^n$ and $c=(c_1,\cdots,c_n)\in\mathbb{C}^n$. By jc we mean $(jc_1,\cdots,jc_n)$ for any $c=(c_1,\cdots,c_n)\in\mathbb{C}^n$ and $j\in\mathbb{N}$. The shift of $f(z)$ is defined by $f(z+c)$, whereas the difference of $f(z)$ is defined by $\Delta_cf(z)=f(z+c)-f(z)$.

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In the past several years, research on various properties and solutions of Fermat-type equations has yielded abundant results and methods (see [6,12,13,16,17]). The equation $f^2+2\alpha fg+g^2=1$ is a generalization of the traditional Fermat-type equation. By introducing the parameter α, a broader range of mathematical structures and properties can be explored. Therefore, more and more researchers begin to study quadratic trinomial equations and they mainly study the existence and exact forms of solutions to this type of equations.

We know that for two meromorphic functions f and g in $\mathbb{C}^n$ functional equation of the form $f^2(z)+2\omega f(z)g(z)+g^2(z)=1$ (where $\omega\neq 0,\pm 1$) is called quadratic trinomial functional equation. In recent years, many scholars paid considerable attention to investigating the existence and form of entire or meromorphic solutions of the type of equations (see [3,8,10,11,15,18]).

In 2013, Saleeby [3] studied the entire solutions of quadratic trinomial Fermat type equation

$$f^2+2\alpha fg+g^2=1 \tag{1.1}$$

and obtained the next Theorem 1.1.

Theorem 1.1. *Let $\alpha^2\neq 0,1$, $\alpha\in\mathbb{C}$. Then the transcendental entire solution of* (1.1) *must be of the form*

$$f=\frac{1}{\sqrt{2}}\left(\frac{\cos(h)}{\sqrt{1+\alpha}}+\frac{\sin(h)}{\sqrt{1-\alpha}}\right),\quad g=\frac{1}{\sqrt{2}}\left(\frac{\cos(h)}{\sqrt{1+\alpha}}-\frac{\sin(h)}{\sqrt{1-\alpha}}\right),$$

where h is an entire function in $\mathbb{C}^n$.

In 2016, Liu et al. [10] studied the existence and the form of solutions of some quadratic trinomial functional equations when $g(z)=f'(z)$ in Eq. (1.1) and obtained the following Theorems 1.2-1.3.

Theorem 1.2. *Equation*

$$f(z)^2+2\alpha f(z)f'(z)+f'(z)^2=1,\quad \alpha^2\neq 1,0,\quad \alpha\in\mathbb{C}, \tag{1.2}$$

has no transcendental meromorphic solutions.

Theorem 1.3. *The finite order transcendental entire solutions of equation*

$$f(z)^2+2\alpha f(z)f(z+c)+f(z+c)^2=1,\quad \alpha^2\neq 1,0,\quad \alpha\in\mathbb{C}, \tag{1.3}$$

must be of order equal to one.

In 2021, Luo et al. [18] replaced the right side of (1.2) and (1.3) by a function $e^{g(z)}$, where g is a polynomial in $\mathbb{C}$ and investigated the transcendental entire solutions with