Analysis in Theory and Applications
Anal. Theory Appl., Vol. **41**, No. 1 (2025), pp. 52-79

DOI: 10.4208/ata.OA-2024-0044
March 2025

On Solutions of Differential-Difference Equations in $\mathbb{C}^n$

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Received 1 November 2024; Accepted (in revised version) 21 February 2025

Abstract. In this paper, we mainly explore the existence of entire solutions of the quadratic trinomial partial differential-difference equation

$$af^2(z)+2\omega f(z)(a_0f(z)+L_{1,2}^{k+s}(f(z)))+b(a_0f(z)+L_{1,2}^{k+s}(f(z)))^2=e^{g(z)}$$

by utilizing Nevanlinna's theory in several complex variables, where $g(z)$ is entire functions in $\mathbb{C}^n$, $\omega\neq 0$ and $a,b,\omega\in\mathbb{C}$. Furthermore, we get the exact froms of solutions of the above differential-difference equation when $\omega=0$. Our results are generalizations of previous results. In addition, some examples are given to illustrate the accuracy of the results.

Key Words: Differential-difference equations, Nevanlinna theory, finite order, entire solutions.

AMS Subject Classifications: 39A45, 30D35, 39A14, 32H30, 35A20

1 Introduction and main results

In this paper, f denotes a meromorphic function in $\mathbb{C}^n$. We assume that the reader is already familiar with the relevent symbols and concepts of Nevanlinna's value distribution theory [1,2,7], such as the proximate function $m(r,f)$, the counting function $N(r,f)$, the reduced counting function $\overline{N}(r,f)$, the characteristic function $T(r,f)$ in $\mathbb{C}^n$, and $S(r,f)$ denotes the quantity satisfying $S(r,f)=o(T(r,f))$ as $r\to\infty$ outside of a possible exceptional set of finite linear measure.

We denote $z+c=(z_1+c_1,\cdots,z_n+c_n)$ for any $z=(z_1,\cdots,z_n)\in\mathbb{C}^n$ and $c=(c_1,\cdots,c_n)\in\mathbb{C}^n$. By jc we mean $(jc_1,\cdots,jc_n)$ for any $c=(c_1,\cdots,c_n)\in\mathbb{C}^n$ and $j\in\mathbb{N}$. The shift of $f(z)$ is defined by $f(z+c)$, whereas the difference of $f(z)$ is defined by $\Delta_c f(z)=f(z+c)-f(z)$.

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In the past several years, research on various properties and solutions of Fermat-type equations has yielded abundant results and methods (see [6,12,13,16,17]). The equation $f^2+2\alpha fg+g^2=1$ is a generalization of the traditional Fermat-type equation. By introducing the parameter α, a broader range of mathematical structures and properties can be explored. Therefore, more and more researchers begin to study quadratic trinomial equations and they mainly study the existence and exact forms of solutions to this type of equations.

We know that for two meromorphic functions f and g in $\mathbb{C}^n$ functional equation of the form $f^2(z)+2\omega f(z)g(z)+g^2(z)=1$ (where $\omega\neq 0,\pm 1$) is called quadratic trinomial functional equation. In recent years, many scholars paid considerable attention to investigating the existence and form of entire or meromorphic solutions of the type of equations (see [3,8,10,11,15,18]).

In 2013, Saleeby [3] studied the entire solutions of quadratic trinomial Fermat type equation

$$f^2+2\alpha fg+g^2=1 \tag{1.1}$$

and obtained the next Theorem 1.1.

Theorem 1.1. *Let $\alpha^2\neq 0,1$, $\alpha\in\mathbb{C}$. Then the transcendental entire solution of* (1.1) *must be of the form*

$$f=\frac{1}{\sqrt{2}}\left(\frac{\cos(h)}{\sqrt{1+\alpha}}+\frac{\sin(h)}{\sqrt{1-\alpha}}\right),\quad g=\frac{1}{\sqrt{2}}\left(\frac{\cos(h)}{\sqrt{1+\alpha}}-\frac{\sin(h)}{\sqrt{1-\alpha}}\right),$$

where h is an entire function in $\mathbb{C}^n$.

In 2016, Liu et al. [10] studied the existence and the form of solutions of some quadratic trinomial functional equations when $g(z)=f'(z)$ in Eq. (1.1) and obtained the following Theorems 1.2-1.3.

Theorem 1.2. *Equation*

$$f(z)^2+2\alpha f(z)f'(z)+f'(z)^2=1,\quad \alpha^2\neq 1,0,\quad \alpha\in\mathbb{C}, \tag{1.2}$$

has no transcendental meromorphic solutions.

Theorem 1.3. *The finite order transcendental entire solutions of equation*

$$f(z)^2+2\alpha f(z)f(z+c)+f(z+c)^2=1,\quad \alpha^2\neq 1,0,\quad \alpha\in\mathbb{C}, \tag{1.3}$$

must be of order equal to one.

In 2021, Luo et al. [18] replaced the right side of (1.2) and (1.3) by a function $e^{g(z)}$, where g is a polynomial in $\mathbb{C}$ and investigated the transcendental entire solutions with finite order of the quadratic trinomial difference equations

$$f(z)^2+2\alpha f(z)f(z+c)+f(z+c)^2=e^{g(z)},$$
$$f(z)^2+2\alpha f(z)f'(z)+f'(z)^2=e^{g(z)},$$
$$f(z+c)^2+2\alpha f(z+c)f'(z)+f'(z)^2=e^{g(z)},$$

where $\alpha^2(\neq 0,1), c$ are constants and $g(z)$ is a polynomial in $\mathbb{C}$.

Later on, in 2022, Zhang et al. [8] further studied above the three equations and established the exact form of finite order transcendental entire solutions of the following Fermat-type trinomial equations

$$f(z)^2+2\alpha f(z)\Delta_c f(z)+\Delta_c f(z)^2=e^{g(z)},$$
$$f(z+c)^2+2\alpha f(z+c)\Delta_c f(z)+\Delta_c f(z)^2=e^{g(z)},$$
$$f'(z)^2+2\alpha f'(z)\Delta_c f(z)+\Delta_c f(z)^2=e^{g(z)},$$

where $\Delta_c f(z)=f(z+c)-f(z)$ and $g(z)$ is a non-constant polynomial in $\mathbb{C}$.

Furthermore, some researchers also studied on entire solutions for several systems of quadratic trinomial Fermat type functional equations in $\mathbb{C}^2$ (see [5,19]).

Recently, some researchers studied the solutions of some quadratic trinomial functional equations in $\mathbb{C}^n$ (see [11,15]).

However, in 2023, Abhijit [12] investigated the following Fermat-type

$$f(z)^2+[a_0 f(z)+\tilde{L_p}^r(f)]^2=1 \tag{1.4}$$

in $\mathbb{C}^3$, where

$$L_1^r(f)=\sum_{j=1}^{k}a_jf(z+jc) \quad \text{and} \quad L_2^r(f)=\sum_{m=1}^{s}b_m\frac{\partial^m f(z+md)}{\partial z_1^m},$$

$\tilde{L_p}^r := L_1^r+L_2^r$ and obtained the following Theorem 1.4.

Theorem 1.4. *Let $c=(c_1,c_2,c_3)$, $d=(d_1,d_2,c_3)$ be two non-zero constants in $\mathbb{C}^3$; a_j, b_m are constants in $\mathbb{C}$ with at least one of a_j or b_m are nonzero, $j=1,2,\cdots,k$, $m=1,2,\cdots,s$, where k,s be positive integers. Then any finite order transcendental entire solutions of* (1.4) *must be one of the following three types:*

(i) If $L_1^r(f)\not\equiv 0$ and $L_2^r(f)\equiv 0$, then

$$f(z)=-i\sinh\Big(L(z)+\sum_{j=1}^{4}H_j(s_j)+\xi\Big),$$

where $L(z)=\sum_{\mu=1}^{3}\alpha_\mu z_\mu$,$H_1(s_1)$ is a polynomial in $s_1:=d_{11}z_1+d_{12}z_2$, $H_2(s_2)$ is a polynomial in $s_2:=d_{22}z_2+d_{23}z_3$, $H_3(s_3)$ is a polynomial in $s_3:=d_{31}z_1+d_{33}z_3$ and

$H_4(s_4)$ is a polynomial in $s_4 := d_{41}z_1 + d_{42}z_2 + d_{43}z_3$ with $d_{11}c_1 + d_{12}c_2 = 0$, $d_{22}c_2 + d_{23}c_3 = 0$, $d_{31}c_1 + d_{33}c_3 = 0$ and $d_{41}c_1 + d_{42}c_2 + d_{43}c_3 = 0$, ξ, α_μ, d_{ij} are all constants in $\mathbb{C}$, and $L(z)$ satisfy relations

$$\begin{cases} a_0 + \sum_{j=1}^{k} a_j e^{jL(c)} = i, \\ a_0 + \sum_{j=1}^{k} a_j e^{-jL(c)} = -i. \end{cases}$$

(ii) *$L_1^r(f) \equiv 0$ and $L_2^r(f) \not\equiv 0$, then*

$$f(z) = -i\sinh\Big(L(z) + \sum_{j=1}^{4} H_j(s_j) + \xi\Big),$$

where $L(z)$ and $H_j(s_j)$ are defined as in (i), $j = 1,2,3,4$, and $L(z)$ satisfies the relation

$$\begin{cases} a_0 + \sum_{m=1}^{s} b_m \alpha^m e^{mL(d)} = i, \\ a_0 + \sum_{m=1}^{s} (-1)^m b_m \alpha^m e^{-mL(d)} = -i, \end{cases}$$

where α can be found from the relation

$$\alpha_1 + d_{11}H_1'(s_1) + d_{31}H_3'(s_3) + d_{41}H_4'(s_4) = \alpha. \tag{1.5}$$

In particular, if $d_{11} \neq 0$, then $H_1(s_1)$ is linear in s_1. If $d_{31} \neq 0$, then $H_3(s_3)$ is linear in s_3 and if $d_{41} \neq 0$, then $H_4(s_4)$ is linear in s_4.

(iii) *If $L_1^r(f) \not\equiv 0$, $L_2^r(f) \not\equiv 0$ with $c = d$, then*

$$f(z) = -i\sinh\Big(L(z) + \sum_{j=1}^{4} H_j(s_j) + \xi\Big),$$

where $L(z)$ and $H_j(s_j)$ are defined as in (i), $j = 1,2,3,4$, and $L(z)$ satisfy relations

$$\begin{cases} a_0 + \sum_{j=1}^{k} a_j e^{-jL(c)} + \sum_{m=1}^{s} (-1)^m b_m \alpha^m e^{-mL(c)} = -i, \\ a_0 + \sum_{j=1}^{k} a_j e^{jL(c)} + \sum_{m=1}^{s} b_m \alpha^m e^{mL(c)} = i, \end{cases}$$

where α satisfies the relation (1.5).

In particular, if $d_{11} \neq 0$, then $H_1(s_1)$ is linear in s_1. If $d_{31} \neq 0$, then $H_3(s_3)$ is linear in s_3 and if $d_{41} \neq 0$, then $H_4(s_4)$ is linear in s_4.

The purpose of this paper is to further generalize Eq. (1.4). If we extend the study of this type of equation to several complex variables, then what would be its solution?

The following question naturally emerge when further studying quadratic trinomial functional equations in $\mathbb{C}^n$.

Question 1. Do solutions exist for equations (1.4) when the constant 1 is replaced by a function $e^{g(z)}$, where $g(z)$ is a polynomial in $\mathbb{C}^n$? Additionally, in this setting, can we find solutions if the binomial equation is replaced by a quadratic trinomial equation in $\mathbb{C}^n$ with arbitrary coefficients?

Inspired by the papers [3,8,10–12,15], in this paper, we mainly explore the complete solutions in $\mathbb{C}^n$ to quadratic trinomial equations that involve combinations of shifts and partial derivatives

$$af^2(z) + 2\omega f(z)(a_0 f(z) + L_{1,2}^{k+s}(f(z))) + b(a_0 f(z) + L_{1,2}^{k+s}(f(z)))^2 = e^{g(z)}, \tag{1.6}$$

where

$$L_{1,2}^{k+s}(f) := L_1^k(f) + L_2^s(f), \quad L_1^k(f) = \sum_{j=1}^{k} a_j f(z+jc) \quad \text{and} \quad L_2^s(f) = \sum_{m=1}^{s} b_m \frac{\partial^m f(z+md)}{\partial z_1^m},$$

a_j, b_m are constants in $\mathbb{C}$ for $j = 1, \cdots, k$; $m = 1, \cdots, s$ and at least one of a_j or b_m are non-zero, $c, d \in \mathbb{C}^n$ as linear shift operator, linear shift partial differential operator, respectively [12].

In this paper, inspired by certain results from [12], by employing Nevanlinna theory for functions of several complex variables, we get the finite order transcendental entire solutions of quadratic trinomial partial differential equations in $\mathbb{C}^n$. Before presenting our results, we define

$$\omega_1 := -\frac{\omega}{\sqrt{ab}} \pm \frac{\sqrt{\omega^2 - ab}}{\sqrt{ab}} \quad \text{and} \quad \omega_2 := -\frac{\omega}{\sqrt{ab}} \mp \frac{\sqrt{\omega^2 - ab}}{\sqrt{ab}},$$

assume that

$$g(z) = \sum_{|I|=0}^{p} a_{\alpha_1, \cdots, \alpha_n} z_1^{\alpha_1} \cdots z_n^{\alpha_n}$$

be polynomial in $\mathbb{C}^n$, where $I = (\alpha_1, \cdots, \alpha_n)$ be two multi-index with $|I| = \sum_{j=0}^{n} \alpha_j$ and α_j are non-negative integers.

We obtain the following result that determines the form of solutions to a quadratic trinomial partial differential-difference Eq. (1.6) in $\mathbb{C}^n$.

Theorem 1.5. *Let $c=(c_1,c_2,\cdots,c_n)$, $d=(d_1,d_2,\cdots,d_n)$ be two nonzero constants in $\mathbb{C}^n$, where n are positive integers, $1\leq i\leq n$ and $a,b,\omega\in\mathbb{C}\setminus\{0\}$. If Eq. (1.6) admits a transcendental entire solution $f:\mathbb{C}^n\to\mathbb{P}^1(\mathbb{C})$ of finite order, then we have one of the following asserts:*

$$f(z)=\frac{\omega_2 e^{g(z)/2+p(z)}-\omega_1 e^{g(z)/2-p(z)}}{\sqrt{a}(\omega_2-\omega_1)},$$

$p(z)$ must be a polynomial of the form $p(z)=L_1(z)+H_1(s)+B_1$, where $L_1(z)=a_{11}z_1+a_{12}z_2+\cdots+a_{1n}z_n$ and $H_1(s)$ are polynomial in $s:=d_1z_1+d_2z_2+\cdots+d_nz_n$ in $\mathbb{C}^n$ with $d_1c_1+d_2c_2+\cdots+d_nc_n=0$ with $H_1(s+c)=H_1(s)$; $a_{11},\cdots,a_{1n},d_1,\cdots,d_n,B_1\in\mathbb{C}$, $g(z)$ must be a polynomial of the form $g(z)=L_2(z)+H_2(s)+B_2$, where $L_2(z)=a_{21}z_1+a_{22}z_2+\cdots+a_{2n}z_n$ and $H_2(s)$ is a polynomial in $s:=d_1z_1+d_2z_2+\cdots+d_nz_n$ in $\mathbb{C}^n$ with $d_1c_1+d_2c_2+\cdots+d_nc_n=0$ and $H_2(s+c)=H_2(s)$; $a_{21},\cdots,a_{2n},d_1,\cdots,d_n,B_2\in\mathbb{C}$, $p(z)$ and $g(z)$ satisfy the following relationships:

(i) If $L_1^k(f)\not\equiv 0$ and $L_2^s(f)\equiv 0$, then

$$\begin{cases}\omega_1\sqrt{b}\sum\limits_{j=1}^{k}a_je^{-L_1(jc)}e^{\frac{L_2(jc)}{2}}+(a_0\omega_1\sqrt{b}-\sqrt{a})=0,\\ \omega_2\sqrt{b}\sum\limits_{j=1}^{k}a_je^{L_1(jc)}e^{\frac{L_2(jc)}{2}}+(a_0\omega_2\sqrt{b}-\sqrt{a})=0.\end{cases}$$

(ii) If $L_1^k(f)\equiv 0$ and $L_2^s(f)\not\equiv 0$, then

$$\begin{cases}\omega_1\sqrt{b}\sum\limits_{m=1}^{s}b_mh_2(z+md)e^{-L_1(md)}e^{\frac{L_2(md)}{2}}+(a_0\omega_1\sqrt{b}-\sqrt{a})=0,\\ \omega_2\sqrt{b}\sum\limits_{m=1}^{s}b_mh_1(z+md)e^{L_1(md)}e^{\frac{L_2(md)}{2}}+(a_0\omega_2\sqrt{b}-\sqrt{a})=0,\end{cases}$$

where

$$\begin{cases}h_1(z)=\left(\dfrac{\partial q_1(z)}{\partial z_i}\right)^{\lambda}+H_{1\lambda}\left(\dfrac{\partial^{\lambda}q_1(z)}{\partial z_i^{\lambda}},\cdots,\dfrac{\partial q_1(z)}{\partial z_i}\right),\\ h_2(z)=\left(\dfrac{\partial q_2(z)}{\partial z_i}\right)^{\lambda}+H_{2\lambda}\left(\dfrac{\partial^{\lambda}q_2(z)}{\partial z_i^{\lambda}},\cdots,\dfrac{\partial q_2(z)}{\partial z_i}\right),\end{cases}$$

$H_{1\lambda}$ is partial differential polynomials of $q_1(z)$ of degree less than λ, $\lambda=1,2\cdots,s$, and similar definition for $H_{2\lambda}$.

(iii) If $L_1^k(f)\not\equiv 0$ and $L_2^s(f)\not\equiv 0$, $c=d$, then

$$\begin{cases} \omega_1\sqrt{b}\sum_{j=1}^{k}a_je^{-L_1(jc)}e^{\frac{L_2(jc)}{2}}+\omega_1\sqrt{b}\sum_{m=1}^{s}b_mh_2(z+mc)e^{-L_1(mc)}e^{\frac{L_2(mc)}{2}} \\ \quad +(a_0\omega_1\sqrt{b}-\sqrt{a})=0, \\ \omega_2\sqrt{b}\sum_{j=1}^{k}a_je^{L_1(jc)}e^{\frac{L_2(jc)}{2}}+\omega_2\sqrt{b}\sum_{m=1}^{s}b_mh_1(z+mc)e^{L_1(mc)}e^{\frac{L_2(mc)}{2}} \\ \quad +(a_0\omega_2\sqrt{b}-\sqrt{a})=0, \end{cases}$$

where $h_1(z)$, $h_2(z)$ are defined in (ii).

The following examples are provided below to confirm the existence and exact form of the solutions of equations in Theorem 1.5.

Example 1.1. Let $a=4$, $\omega=\sqrt{2}$, $b=3$, $p(z)=(5z_1+z_2-z_3)^6+(5z_1+z_2-z_3)^4+\frac{3\pi i}{5}$, $g(z)=z_1-3z_2+5z_3$, $c=(3,(2+\pi i/2),(17+\pi i/2))$ and $H(s)$ is a polynomial in $s:=5z_1+z_2-z_3$. Then, from (i) of Theorem 1.5, we get

$$f(z)=\frac{(-1\mp i\sqrt{5})e^{\Theta(z)}-(-1\pm i\sqrt{5})e^{\Phi(z)}}{\mp 4i\sqrt{5}},$$

where

$$\Theta(z)=\frac{z_1-3z_2+5z_3}{2}+(5z_1+z_2-z_3)^6+(5z_1+z_2-z_3)^4+\frac{3\pi i}{5},$$
$$\Phi(z)=\frac{z_1-3z_2+5z_3}{2}-(5z_1+z_2-z_3)^6-(5z_1+z_2-z_3)^4-\frac{3\pi i}{5},$$

is a transcendental entire solution with $\rho(f)=6>1$ in $\mathbb{C}^3$ of

$$4f^2(z)+2\sqrt{2}f(z)\left(2f(z)+L_1^1(f)\right)+3\left(2f(z)+L_1^1(f)\right)^2=e^{g(z)},$$

where

$$L_1^1(f)=\frac{2\sqrt{4+\sqrt{2}}}{\sqrt{3}}f(z+c).$$

Example 1.2. Let $a=3$, $\omega=-\sqrt{2}$, $b=4$, $p(z)=\pi i/7$, $g(z)=5z_1+3z_2+5z_3$ $d=(1,-1,-1)$, and $H(s)$ is a polynomial in $s:=4z_1+5z_2+2z_3$, and $a_0=\frac{(3-\sqrt{2}\pm i\sqrt{10}(3+4\sqrt{3}))}{2+i\sqrt{10}}$. Then, from (ii) of Theorem 1.5, we get

$$f(z)=\frac{(1\mp i\sqrt{5})e^{\Theta(z)}-(1\pm i\sqrt{5})e^{\Phi(z)}}{\mp 2i\sqrt{15}},$$

where

$$\Theta(z) = \frac{5z_1 + 3z_2 + 5z_3}{z} + \pi i/7, \quad \Phi(z) = \frac{5z_1 + 3z_2 + 5z_3}{z} - \pi i/7,$$

is a transcendental entire solution in $\mathbb{C}^3$ of

$$3f^2(z) - 2\sqrt{2}f(z)L_2^3(f) + 4\left(L_2^3(f)\right)^2 = e^{g(z)},$$

where

$$L_2^3(f) = a_0 f(z) + \frac{\partial f(z+d)}{\partial z_1} + \frac{\partial^2 f(z+2d)}{\partial z_1^2} + \frac{\partial^3 f(z+3d)}{\partial z_1^3}.$$

2 Some lemmas

The following four lemmas will play a key role in proving the main results of the paper.

Lemma 2.1. *For any entire function F on $\mathbb{C}^n$, $F(0) \neq 0$ and put $\rho(n_F) = \rho < \infty$, where $\rho(n_F)$ denotes be the order of the counting function of zeros of F. Then there exist a canonical function f_F and a function $g_F \in \mathbb{C}^n$ such that $F(z) = f_F(z)e^{g_F(z)}$. For the special case $n = 1$, f_F is the canonical product of Weierstrass.*

Lemma 2.1 [9, 14] and Lemma 2.2 [4] are widely used in the proof of many Fermat type functional equations and quadratic trinomial equations.

Lemma 2.2. *If g and h are entire functions on the complex plane $\mathbb{C}$ and $g(h)$ is an entire function of finite order, then there are only two possible case: either*

(i) the internal function h is a polynomial and the external function g is of finite order; or

(ii) the internal function h is not a polynomial but a function of finite order, and the external function g is of zero order.

Lemma 2.3 [7] is mainly to find the expression of $g(z)$ in the proof of Theorem 1.1 and Remark 4.1 in this paper.

Lemma 2.3. *Suppose that $a_0(z), a_1(z), \cdots, a_m(z)$ $(m \geq 1)$ are meromorphic functions of order $\leq \lambda$ on $\mathbb{C}^n$ and $g_0(z), g_1(z), \cdots, g_m(z)$ are entire function on $\mathbb{C}^n$ such that $g_j(z) - g_k(z)$ $(j \neq k)$ are transcendental or polynomials of degree higher than λ. Then we see that*

$$\sum_{j=0}^{n} a_j(z)e^{g_j(z)} \equiv 0$$

holds only when $a_j(z) \equiv 0$ $(j = 0, 1, \cdots, n)$.

3 Proof of Theorem 1.5

Assume that f is a finite order transcendental entire solution of Eq. (1.6). Firstly, it follows from (1.6) that

$$aA^2+2\omega AB+bB^2=1, \tag{3.1}$$

where

$$A=\frac{f(z)}{e^{\frac{g(z)}{2}}} \quad \text{and} \quad B=\frac{a_0f(z)+L_{1,2}^{k+s}(f)}{e^{\frac{g(z)}{2}}}. \tag{3.2}$$

Then, (3.1) can be rewritten as

$$(\sqrt{a}A-\omega_1\sqrt{b}B)(\sqrt{a}A-\omega_2\sqrt{b}B)=1.$$

Noting that f is a finite order transcendental entire function and g is a polynomial, $(\sqrt{a}A-\omega_1\sqrt{b}B)$ and $(\sqrt{a}A-\omega_2\sqrt{b}B)$ are entire functions without zeros. Then, it follows from Lemmas 2.1 and 2.2 that there exists a polynomial $p(z)$ in $\mathbb{C}^n$ such that

$$\sqrt{a}A-\omega_1\sqrt{b}B=e^p \quad \text{and} \quad \sqrt{a}A-\omega_2\sqrt{b}B=e^{-p}. \tag{3.3}$$

Thus, by (3.2) and (3.3), we can obtain

$$f(z)=\frac{\omega_2e^{p(z)}-\omega_1e^{-p(z)}}{\sqrt{a}(\omega_2-\omega_1)}e^{\frac{g(z)}{2}} \tag{3.4}$$

and

$$a_0f(z)+L_{1,2}^{k+s}(f)=\frac{e^{p(z)}-e^{-p(z)}}{\sqrt{b}(\omega_2-\omega_1)}e^{\frac{g(z)}{2}}. \tag{3.5}$$

For convenience, suppose that

$$q_1(z)=\frac{g(z)}{2}+p(z) \quad \text{and} \quad q_2(z)=\frac{g(z)}{2}-p(z). \tag{3.6}$$

Thus, Eqs. (3.4) and (3.5) can be written as

$$f(z)=\frac{\omega_2e^{q_1(z)}-\omega_1e^{q_2(z)}}{\sqrt{a}(\omega_2-\omega_1)} \tag{3.7}$$

and

$$a_0f(z)+L_{1,2}^{k+s}(f)=\frac{e^{q_1(z)}-e^{q_2(z)}}{\sqrt{b}(\omega_2-\omega_1)} \tag{3.8}$$

Next we consider three cases as follows.

Case 1: Suppose that $L_1^k(f) \not\equiv 0$ and $L_2^s(f) \equiv 0$, then, by (3.4) and (3.5), we can get a simple formula as follows

$$\begin{aligned}&\omega_2\sqrt{b}\sum_{j=1}^{k}a_j e^{p(z+jc)+p(z)}e^{\frac{g(z+jc)-g(z)}{2}}-\omega_1\sqrt{b}\sum_{j=1}^{k}a_j e^{p(z)-p(z+jc)}e^{\frac{g(z+jc)-g(z)}{2}}\\&+(a_0\omega_2\sqrt{b}-\sqrt{a})e^{2p(z)}=(a_0\omega_1\sqrt{b}-\sqrt{a}).\end{aligned}\tag{3.9}$$

Next we consider whether $p(z+c)-p(z)$ is constant.

Subcase 1.1: $p(z+c)-p(z)=\eta$, where η is a constant in $\mathbb{C}$. Since $p(z)$ is a polynomial in $\mathbb{C}^n$, then we have

$$p(z)=L_1(z)+H_1(s)+B_1,\tag{3.10}$$

where $L_1(z)=a_{11}z_1+a_{12}z_2+\cdots+a_{1n}z_n$ and $H_1(s)$ is a polynomial in $s:=d_1z_1+d_2z_2+\cdots+d_nz_n$ in $\mathbb{C}^n$ with $d_1c_1+d_2c_2+\cdots+d_nc_n=0$ and $H_1(s+c)=H_1(s)$; $a_{11},\cdots,a_{1n},d_1,\cdots,d_n$, $B_1\in\mathbb{C}$. Hence, we get $p(z+jc)-p(z)=L_1(jc)$ for all $j\in\mathbb{N}$. It follows from (3.9) that

$$\begin{aligned}&\omega_2\sqrt{b}\sum_{j=1}^{k}a_j e^{p(z+jc)+p(z)}e^{\frac{g(z+jc)-g(z)}{2}}+(a_0\omega_2\sqrt{b}-\sqrt{a})e^{2p(z)}\\=&\omega_1\sqrt{b}\sum_{j=1}^{k}a_j e^{-L_1(jc)}e^{\frac{g(z+jc)-g(z)}{2}}+(a_0\omega_1\sqrt{b}-\sqrt{a}).\end{aligned}\tag{3.11}$$

Subcase 1.1.1: Suppose that

$$\omega_1\sqrt{b}\sum_{j=1}^{k}a_j e^{-L_1(jc)}e^{\frac{g(z+jc)-g(z)}{2}}+(a_0\omega_1\sqrt{b}-\sqrt{a})=0,\tag{3.12}$$

then, by (3.11), we get

$$\omega_2\sqrt{b}\sum_{j=1}^{k}a_j e^{L_1(jc)}e^{\frac{g(z+jc)-g(z)}{2}}+(a_0\omega_2\sqrt{b}-\sqrt{a})=0.\tag{3.13}$$

If there exists i $(i=1,\cdots,k)$, such that $g(z+ic)-g(z)$ is a nonconstant, then $g(z+jc)-g(z)$ is a nonconstant for each $j=1,2,\cdots,k$, since a, b, a_j are non-zero and $\omega_1\neq\omega_2$, by applying Lemma 2.3 to (3.12) and (3.13) respectively, we get

$$a_0=\frac{\sqrt{a}}{\omega_2\sqrt{b}}=\frac{\sqrt{a}}{\omega_1\sqrt{b}}\quad\text{and}\quad a_j=0\quad\text{for all}\quad j=1,2,\cdots,k,$$

this is a contradiction to our assumption that $\omega_1 \neq \omega_2$. Hence, there exists i $(i = 1, \cdots, k)$, such that $g(z + ic) - g(z)$ is a constant, then $g(z)$ is a polynomial.

Thus, it follows that $g(z) = L_2(z) + H_2(s) + B_2$, where $L_2(z) = a_{21}z_1 + a_{22}z_2 + \cdots + a_{2n}z_n$ and $H_2(s)$ is a polynomial in $s := d_1z_1 + d_2z_2 + \cdots + d_nz_n$ in $\mathbb{C}^n$ with $d_1c_1 + d_2c_2 + \cdots + d_nc_n = 0$ and $H_2(s + c) = H_2(s)$; $a_{21}, \cdots, a_{2n}, d_1, \cdots, d_n$, $B_2 \in \mathbb{C}$, $p(z)$ and $g(z)$ satisfy the following relationship

$$\begin{cases} \omega_1\sqrt{b}\sum_{j=1}^{k} a_j e^{-L_1(jc)} e^{\frac{L_2(jc)}{2}} + (a_0\omega_1\sqrt{b} - \sqrt{a}) = 0, \\ \omega_2\sqrt{b}\sum_{j=1}^{k} a_j e^{L_1(jc)} e^{\frac{L_2(jc)}{2}} + (a_0\omega_2\sqrt{b} - \sqrt{a}) = 0. \end{cases}$$

Subcase 1.1.2: Assume that

$$\omega_1\sqrt{b}\sum_{j=1}^{k} a_j e^{-L_1(jc)} e^{\frac{g(z+jc)-g(z)}{2}} + (a_0\omega_1\sqrt{b} - \sqrt{a}) \neq 0.$$

Then, it follows from (3.11) that

$$\begin{aligned} &\left(\omega_1\sqrt{b}\sum_{j=1}^{k} a_j e^{-L_1(jc)} e^{\frac{g(z+jc)-g(z)}{2}} + (a_0\omega_1\sqrt{b} - \sqrt{a})\right) e^{-2p(z)} \\ =&\omega_2\sqrt{b}\sum_{j=1}^{k} a_j e^{L_1(jc)} e^{\frac{g(z+jc)-g(z)}{2}} + (a_0\omega_2\sqrt{b} - \sqrt{a}). \end{aligned} \tag{3.14}$$

If $p(z)$ is a nonconstant polynomial, from (3.14) and Lemma 2.3, this is a contradiction.

If $p(z)$ is a constant, then $L_1(jc) = 0$. It follows from (3.14) that

$$\begin{aligned} &\left(e^{-2p(z)}\omega_1\sqrt{b}a_j - \omega_2\sqrt{b}a_j\right) e^{\frac{g(z+jc)-g(z)}{2}} \\ &+ \left(e^{-2p(z)}(a_0\omega_1\sqrt{b} - \sqrt{a}) - (a_0\omega_2\sqrt{b} - \sqrt{a})\right) = 0, \quad (j = 1, \cdots, k), \end{aligned} \tag{3.15}$$

if there exists i $(i = 1, \cdots, k)$, such that $g(z + ic) - g(z)$ is a nonconstant, then $g(z + jc) - g(z)$ is a nonconstant for each $j = 1, 2, \cdots, k$, by applying Lemma 2.3 to (3.15), we have

$$e^{-2p(z)}\omega_1\sqrt{b}a_j - \omega_2\sqrt{b}a_j = 0, \quad (j = 1, \cdots, k),$$

and

$$(a_0\omega_1\sqrt{b} - \sqrt{a})e^{-2p(z)} = a_0\omega_2\sqrt{b} - \sqrt{a},$$

since a, b, a_j $(j = 1, \cdots, k)$ are non-zero, $\omega_1 \neq \omega_2$, then from the first equation above, we have $(e^{-2p(z)}\omega_1 - \omega_2) = 0$, substitute it into the second equation above, we obtain $p(z) = 0$. Now, we substitute $p(z) = 0$ into $(e^{-2p(z)}\omega_1 - \omega_2) = 0$, we have $\omega_1 = \omega_2$, this is a contradiction. Hence, there exists i $(i=1,\cdots,k)$, such that $g(z+ic)-g(z)$ is a constant, then $g(z)$ is a polynomial.

Subcase 1.2: Let $p(z+c)-p(z)$ is nonconstant. Hence, we can easily get $p(z+jc)-p(z)$ is nonconstant for each $j=1,2,\cdots,k$. By applying Lemma 2.3 to (3.9), we discover that

$$a_0=\frac{\sqrt{a}}{\omega_2\sqrt{b}}=\frac{\sqrt{a}}{\omega_1\sqrt{b}} \quad \text{and} \quad a_j=0 \quad \text{for all} \quad j=1,2,\cdots,k.$$

This is a contradiction to our assumption that $\omega_1\neq\omega_2$.

Case 2: If $L_1^k(f)\equiv 0$ and $L_2^s(f)\not\equiv 0$. Differentiating (3.7) λ-th times partially with respect z_i, we can obtain

$$\frac{\partial^\lambda f(z)}{\partial z_i^\lambda}=\frac{\omega_2 h_1(z)e^{q_1(z)}-\omega_1 h_2(z)e^{q_2(z)}}{\sqrt{a}(\omega_2-\omega_1)}, \tag{3.16}$$

where

$$\begin{cases} h_1(z)=\left(\dfrac{\partial q_1(z)}{\partial z_i}\right)^\lambda+H_{1\lambda}\left(\dfrac{\partial^\lambda q_1(z)}{\partial z_i^\lambda},\cdots,\dfrac{\partial q_1(z)}{\partial z_i}\right),\\ h_2(z)=\left(\dfrac{\partial q_2(z)}{\partial z_i}\right)^\lambda+H_{2\lambda}\left(\dfrac{\partial^\lambda q_2(z)}{\partial z_i^\lambda},\cdots,\dfrac{\partial q_2(z)}{\partial z_i}\right), \end{cases} \tag{3.17}$$

$H_{1\lambda}$ is partial differential polynomials of $q_1(z)$ of degree less than λ, $\lambda=1,2,\cdots,s$, and similar definition for $H_{2\lambda}$. By (3.7), (3.8) and (3.16), we can get a simple formula as follows

$$\begin{aligned} &\omega_2\sqrt{b}\sum_{m=1}^{s}b_m h_1(z+md)e^{q_1(z+md)-q_2(z)}-\omega_1\sqrt{b}\sum_{m=1}^{s}b_m h_2(z+md)e^{q_2(z+md)-q_2(z)}\\ &\quad+(a_0\omega_2\sqrt{b}-\sqrt{a})e^{q_1(z)-q_2(z)}=(a_0\omega_1\sqrt{b}-\sqrt{a}). \end{aligned} \tag{3.18}$$

Next we consider whether $p(z+c)-p(z)$ is constant.

Subcase 2.1: $p(z+c)-p(z)=\eta$, where η is a constant in $\mathbb{C}$. Since $p(z)$ is a polynomial in $\mathbb{C}^n$, then we have

$$p(z)=L_1(z)+H_1(s)+B_1,$$

where $L_1(z)$, $H_1(s)$, B_1 are defined in Subcase 1.1. Thus, we get $p(z+md)-p(z)=L_1(md)$ for all $m\in\mathbb{N}$. It follows from (3.18) that

$$\begin{aligned} &\omega_2\sqrt{b}\sum_{m=1}^{s}b_m h_1(z+md)e^{p(z+md)+p(z)}e^{\frac{g(z+md)-g(z)}{2}}+(a_0\omega_2\sqrt{b}-\sqrt{a})e^{2p(z)}\\ =&\omega_1\sqrt{b}\sum_{m=1}^{s}b_m h_2(z+md)e^{-L_1(md)}e^{\frac{g(z+md)-g(z)}{2}}+(a_0\omega_1\sqrt{b}-\sqrt{a}). \end{aligned} \tag{3.19}$$

Subcase 2.1.1: If

$$\omega_1\sqrt{b}\sum_{m=1}^{s} b_m h_2(z+md)e^{-L_1(md)}e^{\frac{g(z+md)-g(z)}{2}} + (a_0\omega_1\sqrt{b}-\sqrt{a}) = 0, \tag{3.20}$$

then, from (3.19), we get

$$\omega_2\sqrt{b}\sum_{m=1}^{s} b_m h_1(z+md)e^{L_1(md)}e^{\frac{g(z+md)-g(z)}{2}} + (a_0\omega_2\sqrt{b}-\sqrt{a}) = 0. \tag{3.21}$$

If there exists i $(i=1,\cdots,s)$, such that $g(z+id)-g(z)$ is a nonconstant, then $g(z+md)-g(z)$ is a nonconstant for each $m=1,2,\cdots,s$, since a, b, b_m are non-zero and $\omega_1\neq\omega_2$, by applying Lemma 2.3 to (3.20) and (3.21) respectively, we get

$$a_0=\frac{\sqrt{a}}{\omega_2\sqrt{b}}=\frac{\sqrt{a}}{\omega_1\sqrt{b}},\quad b_m h_1(z+md)=0,\quad b_m h_2(z+md)=0 \quad \text{for all} \quad m=1,2,\cdots,s,$$

this is a contradiction to our assumption that $\omega_1\neq\omega_2$. Hence, there exists i $(i=1,\cdots,s)$, such that $g(z+id)-g(z)$ is a constant, then $g(z)$ is a polynomial.

Thus, it follows that $g(z)=L_2(z)+H_2(s)+B_2$, where $L_2(z)$, $H_2(s)$, B_2 are defined in Subcase 1.1.1, $p(z)$ and $g(z)$ satisfy the following relationship

$$\begin{cases} \omega_1\sqrt{b}\sum\limits_{m=1}^{s} b_m h_2(z+md)e^{-L_1(md)}e^{\frac{L_2(md)}{2}} + (a_0\omega_1\sqrt{b}-\sqrt{a}) = 0, \\ \omega_2\sqrt{b}\sum\limits_{m=1}^{s} b_m h_1(z+md)e^{L_1(md)}e^{\frac{L_2(md)}{2}} + (a_0\omega_2\sqrt{b}-\sqrt{a}) = 0. \end{cases}$$

Subcase 2.1.2: If

$$\omega_1\sqrt{b}\sum_{m=1}^{s} b_m h_2(z+md)e^{-L_1(md)}e^{\frac{g(z+md)-g(z)}{2}} + (a_0\omega_1\sqrt{b}-\sqrt{a}) \neq 0.$$

Then, it follows from (3.19) that

$$\begin{aligned} &\omega_2\sqrt{b}\sum_{m=1}^{s} b_m h_1(z+md)e^{L_1(md)}e^{\frac{g(z+md)-g(z)}{2}} + (a_0\omega_2\sqrt{b}-\sqrt{a}) \\ =&\left(\omega_1\sqrt{b}\sum_{m=1}^{s} b_m h_2(z+md)e^{-L_1(md)}e^{\frac{g(z+md)-g(z)}{2}} + (a_0\omega_1\sqrt{b}-\sqrt{a})\right)e^{-2p(z)}. \end{aligned} \tag{3.22}$$

If $p(z)$ is a nonconstant polynomial, from (3.22) and Lemma 2.3, this is a contradiction.

If $p(z)$ is a constant, then $L_1(md)=0$, $h_1(z)=h_2(z)$. It follows from (3.22) that

$$\begin{aligned} &\left(e^{-2p(z)}\omega_1\sqrt{b}b_m h_2(z+md)-\omega_2\sqrt{b}b_m h_1(z+md)\right)e^{\frac{g(z+md)-g(z)}{2}} \\ &+\left(e^{-2p(z)}(a_0\omega_1\sqrt{b}-\sqrt{a})-(a_0\omega_2\sqrt{b}-\sqrt{a})\right)=0, \quad (m=1,\cdots,s), \end{aligned} \tag{3.23}$$

if there exists i $(i=1,\cdots,s)$, such that $g(z+id)-g(z)$ is a nonconstant, then $g(z+md)-g(z)$ is a nonconstant for each $m=1,2,\cdots,s$, by applying Lemma 2.3 to (3.23), we have

$$e^{-2p(z)}\omega_1\sqrt{b}b_m h_2(z+md)-\omega_2\sqrt{b}b_m h_1(z+md)=0,\quad (m=1,\cdots,s),$$

and

$$(a_0\omega_1\sqrt{b}-\sqrt{a})e^{-2p(z)}=a_0\omega_2\sqrt{b}-\sqrt{a},$$

since a, b, b_m $(m=1,\cdots,s)$ are non-zero, $\omega_1\neq\omega_2$, then from the first equation above, we have $(e^{-2p(z)}\omega_1-\omega_2)=0$, substitute it into the second equation above, we obtain $p(z)=0$. Now, we substitute $p(z)=0$ into $(e^{-2p(z)}\omega_1-\omega_2)=0$, we have $\omega_1=\omega_2$, this is a contradiction. Hence, there exists i $(i=1,\cdots,s)$, such that $g(z+id)-g(z)$ is a constant, then $g(z)$ is a polynomial.

Subcase 2.2: Assume that $p(z+c)-p(z)$ is nonconstant, then by similar arguments of Subcase 1.2, we can also get a contradiction.

Case 3: Suppose that $L_1^k(f)\not\equiv 0$ and $L_2^s(f)\not\equiv 0$ and $c=d\in\mathbb{C}^n$, then, by (3.4), (3.5) and (3.16), we can get a simple formula as follows

$$\begin{aligned}&\omega_2\sqrt{b}\sum_{j=1}^{k}a_j e^{p(z+jc)+p(z)}e^{\frac{g(z+jc)-g(z)}{2}}-\omega_1\sqrt{b}\sum_{j=1}^{k}a_j e^{p(z)-p(z+jc)}e^{\frac{g(z+jc)-g(z)}{2}}\\&+\omega_2\sqrt{b}\sum_{m=1}^{s}b_m h_1(z+mc)e^{p(z+mc)+p(z)}e^{\frac{g(z+mc)-g(z)}{2}}\\&-\omega_1\sqrt{b}\sum_{m=1}^{s}b_m h_2(z+mc)e^{p(z)-p(z+mc)}e^{\frac{g(z+mc)-g(z)}{2}}\\&+(a_0\omega_2\sqrt{b}-\sqrt{a})e^{2p(z)}=(a_0\omega_1\sqrt{b}-\sqrt{a}).\end{aligned}\tag{3.24}$$

Next we consider whether $p(z+c)-p(z)$ is constant.

Subcase 3.1: $p(z+c)-p(z)=\eta$, where η is a constant in $\mathbb{C}$. Since $p(z)$ is a polynomial in $\mathbb{C}^n$, then we have

$$p(z)=L_1(z)+H_1(s)+B_1,$$

where $L_1(z),H_1(s),B_1$ are defined in Subcase 1.1. Thus, we get

$$p(z+jc)-p(z)=L_1(jc),\quad p(z+mc)-p(z)=L_1(mc)$$

for all $j,m\in\mathbb{N}$. It follows from (3.24) that

$$\begin{aligned}&\omega_2\sqrt{b}\sum_{j=1}^{k}a_je^{p(z+jc)+p(z)}e^{\frac{g(z+jc)-g(z)}{2}}+\omega_2\sqrt{b}\sum_{m=1}^{s}b_mh_1(z+mc)e^{p(z+mc)+p(z)}e^{\frac{g(z+mc)-g(z)}{2}}\\&\quad+(a_0\omega_2\sqrt{b}-\sqrt{a})e^{2p(z)}\\=&\omega_1\sqrt{b}\sum_{j=1}^{k}a_je^{-L_1(jc)}e^{\frac{g(z+jc)-g(z)}{2}}+\omega_1\sqrt{b}\sum_{m=1}^{s}b_mh_2(z+mc)e^{-L_1(mc)}e^{\frac{g(z+mc)-g(z)}{2}}\\&\quad+(a_0\omega_1\sqrt{b}-\sqrt{a}).\end{aligned}\tag{3.25}$$

Subcase 3.1.1: If

$$\begin{aligned}&\omega_1\sqrt{b}\sum_{j=1}^{k}a_je^{-L_1(jc)}e^{\frac{g(z+jc)-g(z)}{2}}+\omega_1\sqrt{b}\sum_{m=1}^{s}b_mh_2(z+mc)e^{-L_1(mc)}e^{\frac{g(z+mc)-g(z)}{2}}\\&\quad+(a_0\omega_1\sqrt{b}-\sqrt{a})=0,\end{aligned}\tag{3.26}$$

then, it follows from (3.25) that

$$\begin{aligned}&\omega_2\sqrt{b}\sum_{j=1}^{k}a_je^{L_1(jc)}e^{\frac{g(z+jc)-g(z)}{2}}+\omega_2\sqrt{b}\sum_{m=1}^{s}b_mh_1(z+mc)e^{L_1(mc)}e^{\frac{g(z+mc)-g(z)}{2}}\\&\quad+(a_0\omega_2\sqrt{b}-\sqrt{a})=0.\end{aligned}\tag{3.27}$$

If there exists i $(i\in\mathbb{N})$, such that $g(z+ic)-g(z)$ is nonconstant, then $g(z+jc)-g(z)$ and $g(z+mc)-g(z)$ are nonconstant for each $j,m\in\mathbb{N}$, since a,b are non-zero and at least one of a_j or b_m are non-zero, $\omega_1\neq\omega_2$, by applying Lemma 2.3 to (3.26) and (3.27) respectively, we get

$$a_0=\frac{\sqrt{a}}{\omega_2\sqrt{b}}=\frac{\sqrt{a}}{\omega_1\sqrt{b}},\quad a_j=0,\quad b_mh_1(z+mc)=0,\quad b_mh_2(z+mc)=0$$

for all $j=1,2,\cdots,k$, $m=1,2,\cdots,s$, this is a contradiction to our assumption that $\omega_1\neq\omega_2$. Hence, there exists i $(i\in\mathbb{N})$, such that $g(z+ic)-g(z)$ is a constant, then $g(z)$ is a polynomial.

Thus, it follows that $g(z)=L_2(z)+H_2(s)+B_2$, where $L_2(z)$, $H_2(s)$, B_2 are defined in Subcase 1.1.1, $p(z)$ and $g(z)$ satisfy the following relationship

$$\begin{cases}\omega_1\sqrt{b}\sum_{j=1}^{k}a_je^{-L_1(jc)}e^{\frac{L_2(jc)}{2}}+\omega_1\sqrt{b}\sum_{m=1}^{s}b_mh_2(z+mc)e^{-L_1(mc)}e^{\frac{L_2(mc)}{2}}+(a_0\omega_1\sqrt{b}-\sqrt{a})=0,\\ \omega_2\sqrt{b}\sum_{j=1}^{k}a_je^{L_1(jc)}e^{\frac{L_2(jc)}{2}}+\omega_2\sqrt{b}\sum_{m=1}^{s}b_mh_1(z+mc)e^{L_1(mc)}e^{\frac{L_2(mc)}{2}}+(a_0\omega_2\sqrt{b}-\sqrt{a})=0.\end{cases}$$

Subcase 3.1.2: If

$$\begin{aligned}&\omega_1\sqrt{b}\sum_{j=1}^{k}a_je^{-L_1(jc)}e^{\frac{g(z+jc)-g(z)}{2}}+\omega_1\sqrt{b}\sum_{m=1}^{s}b_mh_2(z+mc)e^{-L_1(mc)}e^{\frac{g(z+mc)-g(z)}{2}}\\&\quad+(a_0\omega_1\sqrt{b}-\sqrt{a})\neq 0.\end{aligned}$$

Then, it follows from (3.25) that

$$\begin{aligned}&\omega_2\sqrt{b}\sum_{j=1}^{k}a_je^{L_1(jc)}e^{g_1(z)}+\omega_2\sqrt{b}\sum_{m=1}^{s}b_mh_1(z+mc)e^{L_1(mc)}e^{g_2(z)}+(a_0\omega_2\sqrt{b}-\sqrt{a})\\=&\Big(\omega_1\sqrt{b}\sum_{j=1}^{k}a_je^{-L_1(jc)}e^{g_1(z)}+\omega_1\sqrt{b}\sum_{m=1}^{s}b_mh_2(z+mc)e^{-L_1(mc)}e^{g_2(z)}\\&\quad+(a_0\omega_1\sqrt{b}-\sqrt{a})\Big)e^{-2p(z)},\end{aligned}\tag{3.28}$$

where

$$g_1(z)=\frac{g(z+jc)-g(z)}{2}\quad\text{and}\quad g_2(z)=\frac{g(z+mc)-g(z)}{2}.$$

If $p(z)$ is a nonconstant polynomial, from (3.28) and Lemma 2.3, this is a contradiction. If $p(z)$ is a constant, then $L_1(mc)=L_1(jc)=0$, $h_1(z)=h_2(z)$.

When $k=s$, it follows from (3.28) that

$$\begin{aligned}&\Big(e^{-2p(z)}\omega_1\sqrt{b}a_j-\omega_2\sqrt{b}a_j+e^{-2p(z)}\omega_1\sqrt{b}b_jh_2(z+jc)-\omega_2\sqrt{b}b_jh_1(z+jc)\Big)e^{\frac{g(z+jc)-g(z)}{2}}\\&\quad+\Big(e^{-2p(z)}(a_0\omega_1\sqrt{b}-\sqrt{a})-(a_0\omega_2\sqrt{b}-\sqrt{a})\Big)=0,\quad (j=1,\cdots,k),\end{aligned}\tag{3.29}$$

if there exists i $(i=1,\cdots,k)$, such that $g(z+ic)-g(z)$ is a nonconstant, then $g(z+jc)-g(z)$ is a nonconstant for each $j=1,2,\cdots,k$, by applying Lemma 2.3 to (3.29), we have

$$\begin{aligned}&e^{-2p(z)}\omega_1\sqrt{b}a_j-\omega_2\sqrt{b}a_j+e^{-2p(z)}\omega_1\sqrt{b}b_jh_2(z+jc)\\&\quad-\omega_2\sqrt{b}b_jh_1(z+jc)=0,\quad (j=1,\cdots,k),\end{aligned}\tag{3.30}$$

and

$$(a_0\omega_1\sqrt{b}-\sqrt{a})e^{-2p(z)}=a_0\omega_2\sqrt{b}-\sqrt{a},\tag{3.31}$$

(3.30) can be written as

$$e^{-2p(z)}\omega_1\sqrt{b}[a_j+b_jh_2(z+jc)]=\omega_2\sqrt{b}[a_j+b_jh_1(z+jc)],\quad (j=1,\cdots,k),$$

since a,b are non-zero and at least one of a_j or b_m are non-zero, $\omega_1\neq\omega_2$, we have $(e^{-2p(z)}\omega_1-\omega_2)=0$, substitute it into (3.31), we obtain $p(z)=0$. Now, we substitute

$p(z)=0$ into $(e^{-2p(z)}\omega_1-\omega_2)=0$, we have $\omega_1=\omega_2$, this is a contradiction. Hence, there exists i $(i=1,\cdots,k)$, such that $g(z+ic)-g(z)$ is a constant, then $g(z)$ is a polynomial.

When $k\neq s$, then $k>s$ or $k<s$.

When $k>s$, it follows from (3.28) that

$$\begin{aligned}&\left(e^{-2p(z)}\omega_1\sqrt{b}\sum_{j=1}^{s}a_j-\omega_2\sqrt{b}\sum_{j=1}^{s}a_j+e^{-2p(z)}\omega_1\sqrt{b}\sum_{j=1}^{s}b_jh_2(z+jc)\right.\\&\left.-\omega_2\sqrt{b}\sum_{j=1}^{s}b_jh_1(z+jc)\right)e^{\frac{g(z+jc)-g(z)}{2}}+\left(e^{-2p(z)}\omega_1\sqrt{b}\sum_{j=s+1}^{k}a_j-\omega_2\sqrt{b}\sum_{j=s+1}^{k}a_j\right)e^{\frac{g(z+jc)-g(z)}{2}}\\&+\Big(e^{-2p(z)}(a_0\omega_1\sqrt{b}-\sqrt{a})-(a_0\omega_2\sqrt{b}-\sqrt{a})\Big)=0,\end{aligned}\tag{3.32}$$

if there exists i $(i=1,\cdots,k)$, such that $g(z+ic)-g(z)$ is a nonconstant, then $g(z+jc)-g(z)$ is a nonconstant for each $j=1,2,\cdots,k$, by applying Lemma 2.3 to (3.32), we have

$$\begin{aligned}&e^{-2p(z)}\omega_1\sqrt{b}a_j-\omega_2\sqrt{b}a_j+e^{-2p(z)}\omega_1\sqrt{b}b_jh_2(z+jc)\\&\quad-\omega_2\sqrt{b}b_jh_1(z+jc)=0,\quad (j=1,\cdots,s),\end{aligned}\tag{3.33}$$

and

$$e^{-2p(z)}\omega_1\sqrt{b}a_j-\omega_2\sqrt{b}a_j=0,\quad (j=s+1,\cdots,k),\tag{3.34}$$

and

$$(a_0\omega_1\sqrt{b}-\sqrt{a})e^{-2p(z)}=a_0\omega_2\sqrt{b}-\sqrt{a},\tag{3.35}$$

(3.33) can be written as

$$e^{-2p(z)}\omega_1\sqrt{b}[a_j+b_jh_2(z+jc)]=\omega_2\sqrt{b}[a_j+b_jh_1(z+jc)],\quad (j=1,\cdots,s),\tag{3.36}$$

since a, b are non-zero and at least one of a_j or b_m are non-zero, $\omega_1\neq\omega_2$, from (3.34) and (3.36), we all have $(e^{-2p(z)}\omega_1-\omega_2)=0$, substitute it into (3.35), we obtain $p(z)=0$. Now, we substitute $p(z)=0$ into $(e^{-2p(z)}\omega_1-\omega_2)=0$, we have $\omega_1=\omega_2$, this is a contradiction. Hence, there exists i $(i=1,\cdots,k)$, such that $g(z+ic)-g(z)$ is a constant, then $g(z)$ is a polynomial. Similarly, when $k<s$, we can also get a contradiction.

Subcase 3.2: Suppose that $p(z+c)-p(z)$ is nonconstant. Hence, we can easily get $p(z+jc)-p(z)$ is nonconstant for each $j=1,2,\cdots,k$ and $p(z+mc)-p(z)$ is nonconstant for each $m=1,2,\cdots,s$. By applying Lemma 2.3 to (3.24), we discover that

$$\begin{aligned}&a_0=\frac{\sqrt{a}}{\omega_2\sqrt{b}}=\frac{\sqrt{a}}{\omega_1\sqrt{b}},\quad b_mh_1(z+mc)=0,\quad b_mh_2(z+mc)=0\\&\text{and}\quad a_j=0\quad\text{for all }\ j=1,2,\cdots,k.\end{aligned}$$

This is a contradiction to our assumption that $\omega_1\neq\omega_2$. This completes the proof of Theorem 1.5. □

4 Further discussion

In particular, if $\omega = 0$, then the difference Eq. (1.6) in $\mathbb{C}^n$ can be rewritten as

$$af^2(z) + b(a_0 f(z) + L_{1,2}^{k+s}(f(z)))^2 = e^{g(z)}. \tag{4.1}$$

We discuss the finite order solutions of (4.1) in the following result.

Theorem 4.1. *Let $c = (c_1, c_2, \cdots, c_n)$, $d = (d_1, d_2, \cdots, d_n)$ be two nonzero constants in $\mathbb{C}^n$, where n are positive integer, $1 \leq i \leq n$ and $a, b \in \mathbb{C} \setminus \{0\}$. If Eq. (4.1) admits a transcendental entire solution $f : \mathbb{C}^n \to \mathbb{P}^1(\mathbb{C})$ of finite order, then we have one of the following asserts:*

$$f(z) = \frac{e^{g(z)/2+p(z)} + e^{g(z)/2-p(z)}}{2\sqrt{a}},$$

$p(z)$ must be a polynomial of the form $p(z) = L_1(z) + H_1(s) + B_1$, where $L_1(z) = a_{11}z_1 + a_{12}z_2 + \cdots + a_{1n}z_n$ and $H_1(s)$ are polynomial in $s := d_1z_1 + d_2z_2 + \cdots + d_nz_n$ in $\mathbb{C}^n$ with $d_1c_1 + d_2c_2 + \cdots + d_nc_n = 0$ with $H_1(s + c) = H_1(s)$; $a_{11}, \cdots, a_{1n}, d_1, \cdots, d_n, B_1 \in \mathbb{C}$, $g(z)$ must be a polynomial of the form $g(z) = L_2(z) + H_2(s) + B_2$, where $L_2(z) = a_{21}z_1 + a_{22}z_2 + \cdots + a_{2n}z_n$ and $H_2(s)$ is a polynomial in $s := d_1z_1 + d_2z_2 + \cdots + d_nz_n$ in $\mathbb{C}^n$ with $d_1c_1 + d_2c_2 + \cdots + d_nc_n = 0$ and $H_2(s + c) = H_2(s)$; $a_{21}, \cdots, a_{2n}, d_1, \cdots, d_n, B_2 \in \mathbb{C}$, $p(z)$ and $g(z)$ satisfy the following relationships:

(i) If $L_1^k(f) \not\equiv 0$ and $L_2^s(f) \equiv 0$, then

$$\begin{cases} i\sqrt{b}\displaystyle\sum_{j=1}^{k} a_j e^{-L_1(jc)} e^{\frac{L_2(jc)}{2}} + (ia_0\sqrt{b} + \sqrt{a}) = 0, \\ i\sqrt{b}\displaystyle\sum_{j=1}^{k} a_j e^{L_1(jc)} e^{\frac{L_2(jc)}{2}} + (ia_0\sqrt{b} - \sqrt{a}) = 0. \end{cases}$$

(ii) If $L_1^k(f) \equiv 0$ and $L_2^s(f) \not\equiv 0$, then

$$\begin{cases} i\sqrt{b}\displaystyle\sum_{m=1}^{s} b_m h_2(z + md) e^{-L_1(md)} e^{\frac{L_2(md)}{2}} + (ia_0\sqrt{b} + \sqrt{a}) = 0, \\ i\sqrt{b}\displaystyle\sum_{m=1}^{s} b_m h_1(z + md) e^{L_1(md)} e^{\frac{L_2(md)}{2}} + (ia_0\sqrt{b} - \sqrt{a}) = 0, \end{cases}$$

where

$$\begin{cases} h_1(z) = \left(\dfrac{\partial q_1(z)}{\partial z_i}\right)^{\lambda} + H_{1\lambda}\left(\dfrac{\partial^{\lambda} q_1(z)}{\partial z_i^{\lambda}}, \cdots, \dfrac{\partial q_1(z)}{\partial z_i}\right), \\ h_2(z) = \left(\dfrac{\partial q_2(z)}{\partial z_i}\right)^{\lambda} + H_{2\lambda}\left(\dfrac{\partial^{\lambda} q_2(z)}{\partial z_i^{\lambda}}, \cdots, \dfrac{\partial q_2(z)}{\partial z_i}\right), \end{cases}$$

$H_{1\lambda}$ is partial differential polynomials of $q_1(z)$ of degree less than λ, $\lambda=1,2\cdots,s$, and similar definition for $H_{2\lambda}$.

(iii) If $L_1^k(f)\not\equiv 0$ and $L_2^s(f)\not\equiv 0$, $c=d$, then

$$\begin{cases} i\sqrt{b}\sum_{j=1}^{k} a_j e^{-L_1(jc)} e^{\frac{L_2(jc)}{2}} + i\sqrt{b}\sum_{m=1}^{s} b_m h_2(z+md) e^{-L_1(mc)} e^{\frac{L_2(mc)}{2}} + (ia_0\sqrt{b}+\sqrt{a}) = 0, \\ i\sqrt{b}\sum_{j=1}^{k} a_j e^{L_1(jc)} e^{\frac{L_2(jc)}{2}} + i\sqrt{b}\sum_{m=1}^{s} b_m h_1(z+md) e^{L_1(mc)} e^{\frac{L_2(mc)}{2}} + (ia_0\sqrt{b}-\sqrt{a}) = 0, \end{cases}$$

where $h_1(z)$, $h_2(z)$ are defined in (ii).

Remark 4.1. It is easy to find out that Theorem 4.1 becomes Theorem 1.4 in $\mathbb{C}^3$ When $a=b=1$ and $g(z)=0$ in Eq. (4.1). So that means Eq. (1.6) generalizes Eq. (1.4).

Proof of Theorem 4.1. Let f is a finite order transcendental entire solution of Eq. (4.1). First we write (4.1) as following

$$\left[\frac{\sqrt{a}f(z)}{e^{\frac{g(z)}{2}}} + i\frac{\sqrt{b}(a_0 f(z)+L_{1,2}^{k+s}(f))}{e^{\frac{g(z)}{2}}}\right]\left[\frac{\sqrt{a}f(z)}{e^{\frac{g(z)}{2}}} - i\frac{\sqrt{b}(a_0 f(z)+L_{1,2}^{k+s}(f))}{e^{\frac{g(z)}{2}}}\right] = 1.$$

Since f is a finite order transcendental entire function and g is a polynomial. Then, it follows from Lemmas 2.1 and 2.2 that there exists a polynomial $p(z)$ in $\mathbb{C}^n$ such that

$$\frac{\sqrt{a}f(z)}{e^{\frac{g(z)}{2}}} + i\frac{\sqrt{b}(a_0 f(z)+L_{1,2}^{k+s}(f))}{e^{\frac{g(z)}{2}}} = e^{p}, \tag{4.2a}$$

$$\frac{\sqrt{a}f(z)}{e^{\frac{g(z)}{2}}} - i\frac{\sqrt{b}(a_0 f(z)+L_{1,2}^{k+s}(f))}{e^{\frac{g(z)}{2}}} = e^{-p}. \tag{4.2b}$$

Thus, by (4.2), we obtain that

$$f(z) = \frac{e^{p(z)}+e^{-p(z)}}{2\sqrt{a}} e^{\frac{g(z)}{2}} \tag{4.3}$$

and

$$a_0 f(z) + L_{1,2}^{k+s}(f) = \frac{e^{p(z)}-e^{-p(z)}}{2i\sqrt{b}} e^{\frac{g(z)}{2}}. \tag{4.4}$$

For convenience, suppose that

$$q_1(z) = \frac{g(z)}{2} + p(z) \quad \text{and} \quad q_2(z) = \frac{g(z)}{2} - p(z). \tag{4.5}$$

Thus, Eqs. (4.3) and (4.4) can be written as

$$f(z)=\frac{e^{q_1(z)}+e^{q_2(z)}}{2\sqrt{a}} \tag{4.6}$$

and

$$a_0f(z)+L_{1,2}^{k+s}(f)=\frac{e^{q_1(z)}-e^{q_2(z)}}{2i\sqrt{b}}. \tag{4.7}$$

Next we consider three cases as follows.

Case 1: Suppose that $L_1^k(f)\not\equiv 0$ and $L_2^s(f)\equiv 0$, then, by (4.3) and (4.4), we can get a simple formula as follows

$$\begin{aligned}&i\sqrt{b}\sum_{j=1}^{k}a_je^{p(z+jc)+p(z)}e^{\frac{g(z+jc)-g(z)}{2}}+i\sqrt{b}\sum_{j=1}^{k}a_je^{p(z)-p(z+jc)}e^{\frac{g(z+jc)-g(z)}{2}}\\&+(ia_0\sqrt{b}-\sqrt{a})e^{2p(z)}=(-ia_0\sqrt{b}-\sqrt{a}).\end{aligned} \tag{4.8}$$

Next we consider whether $p(z+c)-p(z)$ is constant.

Subcase 1.1: $p(z+c)-p(z)=\eta$, where η is a constant in $\mathbb{C}$. Since $p(z)$ is a polynomial in $\mathbb{C}^n$, then we have

$$p(z)=L_1(z)+H_1(s)+B_1, \tag{4.9}$$

where $L_1(z)=a_{11}z_1+a_{12}z_2+\cdots+a_{1n}z_n$ and $H_1(s)$ is a polynomial in $s:=d_1z_1+d_2z_2+\cdots+d_nz_n$ in $\mathbb{C}^n$ with $d_1c_1+d_2c_2+\cdots+d_nc_n=0$ and $H_1(s+c)=H_1(s)$; $a_{11},\cdots,a_{1n},d_1,\cdots,d_n$, $B_1\in\mathbb{C}$. Hence, we get $p(z+jc)-p(z)=L_1(jc)$ for all $j\in\mathbb{N}$. It follows from (4.8) that

$$\begin{aligned}&i\sqrt{b}\sum_{j=1}^{k}a_je^{p(z+jc)+p(z)}e^{\frac{g(z+jc)-g(z)}{2}}+(ia_0\sqrt{b}-\sqrt{a})e^{2p(z)}\\=&-i\sqrt{b}\sum_{j=1}^{k}a_je^{-L_1(jc)}e^{\frac{g(z+jc)-g(z)}{2}}+(-ia_0\sqrt{b}-\sqrt{a}).\end{aligned} \tag{4.10}$$

Subcase 1.1.1: Suppose that

$$-i\sqrt{b}\sum_{j=1}^{k}a_je^{-L_1(jc)}e^{\frac{g(z+jc)-g(z)}{2}}+(-ia_0\sqrt{b}-\sqrt{a})=0, \tag{4.11}$$

then, by (4.10), we get

$$i\sqrt{b}\sum_{j=1}^{k}a_je^{L_1(jc)}e^{\frac{g(z+jc)-g(z)}{2}}+(ia_0\sqrt{b}-\sqrt{a})=0. \tag{4.12}$$

If there exists i $(i = 1,\cdots,k)$, such that $g(z+ic)-g(z)$ is a nonconstant, then $g(z+jc)-g(z)$ is a nonconstant for each $j=1,2,\cdots,k$, since a, b, a_j are non-zero, by applying Lemma 2.3 to (4.11) and (4.12) respectively, we get $ia_0\sqrt{b}+\sqrt{a}=ia_0\sqrt{b}-\sqrt{a}=0$ and $a_j=0$ for all $j=1,2,\cdots,k$, this is a contradiction to our assumption that $a\neq 0$. Hence, there exists i $(i=1,\cdots,k)$, such that $g(z+ic)-g(z)$ is a constant, then $g(z)$ is a polynomial.

Thus, it follows that $g(z)=L_2(z)+H_2(s)+B_2$, where $L_2(z)=a_{21}z_1+a_{22}z_2+\cdots+a_{2n}z_n$ and $H_2(s)$ is a polynomial in $s:=d_1z_1+d_2z_2+\cdots+d_nz_n$ in $\mathbb{C}^n$ with $d_1c_1+d_2c_2+\cdots+d_nc_n=0$ and $H_2(s+c)=H_2(s)$; $a_{21},\cdots,a_{2n},d_1,\cdots,d_n,B_2\in\mathbb{C}$, $p(z)$ and $g(z)$ satisfy the following relationship

$$\begin{cases} i\sqrt{b}\sum_{j=1}^{k}a_je^{-L_1(jc)}e^{\frac{L_2(jc)}{2}}+(ia_0\sqrt{b}+\sqrt{a})=0,\\ i\sqrt{b}\sum_{j=1}^{k}a_je^{L_1(jc)}e^{\frac{L_2(jc)}{2}}+(ia_0\sqrt{b}-\sqrt{a})=0. \end{cases}$$

Subcase 1.1.2: Assume that

$$-i\sqrt{b}\sum_{j=1}^{k}a_je^{-L_1(jc)}e^{\frac{g(z+jc)-g(z)}{2}}+(-ia_0\sqrt{b}-\sqrt{a})\neq 0.$$

Then, it follows from (4.10) that

$$\begin{aligned}&\left(-i\sqrt{b}\sum_{j=1}^{k}a_je^{-L_1(jc)}e^{\frac{g(z+jc)-g(z)}{2}}+(-ia_0\sqrt{b}-\sqrt{a})\right)e^{-2p(z)}\\ =&i\sqrt{b}\sum_{j=1}^{k}a_je^{L_1(jc)}e^{\frac{g(z+jc)-g(z)}{2}}+(ia_0\sqrt{b}-\sqrt{a}).\end{aligned}\tag{4.13}$$

If $p(z)$ is a nonconstant polynomial, from (4.13) and Lemma 2.3, this is a contradiction.

If $p(z)$ is a constant, then $L_1(jc)=0$. It follows from (4.13) that

$$\begin{aligned}&\left(-ie^{-2p(z)}\sqrt{b}a_j-i\sqrt{b}a_j\right)e^{\frac{g(z+jc)-g(z)}{2}}\\ &\quad+\left(e^{-2p(z)}(-ia_0\sqrt{b}-\sqrt{a})-(ia_0\sqrt{b}-\sqrt{a})\right)=0,\quad (j=1,\cdots,k),\end{aligned}\tag{4.14}$$

if there exists i $(i=1,\cdots,k)$, such that $g(z+ic)-g(z)$ is a nonconstant, then $g(z+jc)-g(z)$ is a nonconstant for each $j=1,2,\cdots,k$, by applying Lemma 2.3 to (4.14), we have

$$-ie^{-2p(z)}\sqrt{b}a_j-i\sqrt{b}a_j=0,\quad (j=1,\cdots,k),$$

and

$$e^{-2p(z)}(-ia_0\sqrt{b}-\sqrt{a})=ia_0\sqrt{b}-\sqrt{a},$$

since a, b, a_j, $(j=1,\cdots,k)$ are non-zero, then from the first equation above, we have $(e^{-2p(z)}+1)=0$, substitute it into the second equation above, we obtain $(e^{-2p(z)}-1)=0$, this is a contradiction. Hence, there exists i $(i=1,\cdots,k)$, such that $g(z+ic)-g(z)$ is a constant, then $g(z)$ is a polynomial.

Subcase 1.2: Let $p(z+c)-p(z)$ is nonconstant. Hence, we can easily get $p(z+jc)-p(z)$ is nonconstant for each $j=1,2,\cdots,k$. By applying Lemma 2.3 to (4.8), we discover that

$$ia_0\sqrt{b}+\sqrt{a}=ia_0\sqrt{b}-\sqrt{a}=0 \quad \text{and} \quad a_j=0 \quad \text{for all} \quad j=1,2,\cdots,k.$$

This is a contradiction to our assumption that $a\neq 0$.

Case 2: If $L_1^k(f)\equiv 0$ and $L_2^s(f)\not\equiv 0$. Differentiating (4.6) λ-th times partially with respect z_i, we can obtain

$$\frac{\partial^\lambda f(z)}{\partial z_i^\lambda}=\frac{h_1(z)e^{q_1(z)}+h_2(z)e^{q_2(z)}}{2\sqrt{a}}, \tag{4.15}$$

where

$$\begin{cases} h_1(z)=\left(\dfrac{\partial q_1(z)}{\partial z_i}\right)^\lambda+H_{1\lambda}\left(\dfrac{\partial^\lambda q_1(z)}{\partial z_i^\lambda},\cdots,\dfrac{\partial q_1(z)}{\partial z_i}\right), \\ h_2(z)=\left(\dfrac{\partial q_2(z)}{\partial z_i}\right)^\lambda+H_{2\lambda}\left(\dfrac{\partial^\lambda q_2(z)}{\partial z_i^\lambda},\cdots,\dfrac{\partial q_2(z)}{\partial z_i}\right), \end{cases} \tag{4.16}$$

$H_{1\lambda}$ is partial differential polynomials of $q_1(z)$ of degree less than λ, $\lambda=1,2\cdots,s$, and similar definition for $H_{2\lambda}$. By (4.6), (4.7) and (4.15), we can get a simple formula as follows

$$\begin{aligned} &i\sqrt{b}\sum_{m=1}^{s}b_m h_1(z+md)e^{q_1(z+md)-q_2(z)}+i\sqrt{b}\sum_{m=1}^{s}b_m h_2(z+md)e^{q_2(z+md)-q_2(z)} \\ &\quad+(ia_0\sqrt{b}-\sqrt{a})e^{q_1(z)-q_2(z)}=(-ia_0\sqrt{b}-\sqrt{a}). \end{aligned} \tag{4.17}$$

Next we consider whether $p(z+c)-p(z)$ is constant.

Subcase 2.1: $p(z+c)-p(z)=\eta$, where η is a constant in $\mathbb{C}$. Since $p(z)$ is a polynomial in $\mathbb{C}^n$, then we have

$$p(z)=L_1(z)+H_1(s)+B_1,$$

where $L_1(z)$, $H_1(s)$, B_1 are defined in Subcase 1.1. Thus, we get $p(z+md)-p(z)=L_1(md)$ for all $m\in\mathbb{N}$. It follows from (4.17) that

$$\begin{aligned} &i\sqrt{b}\sum_{m=1}^{s}b_m h_1(z+md)e^{p(z+md)+p(z)}e^{\frac{g(z+md)-g(z)}{2}}+(ia_0\sqrt{b}-\sqrt{a})e^{2p(z)} \\ =&-i\sqrt{b}\sum_{m=1}^{s}b_m h_2(z+md)e^{-L_1(md)}e^{\frac{g(z+md)-g(z)}{2}}+(-ia_0\sqrt{b}-\sqrt{a}). \end{aligned} \tag{4.18}$$

Subcase 2.1.1: If

$$-i\sqrt{b}\sum_{m=1}^{s} b_m h_2(z+md)e^{-L_1(md)}e^{\frac{g(z+md)-g(z)}{2}}+(-ia_0\sqrt{b}-\sqrt{a})=0, \tag{4.19}$$

then, from (4.18), we get

$$i\sqrt{b}\sum_{m=1}^{s} b_m h_1(z+md)e^{L_1(md)}e^{\frac{g(z+md)-g(z)}{2}}+(ia_0\sqrt{b}-\sqrt{a})=0. \tag{4.20}$$

If there exists i $(i=1,\cdots,s)$, such that $g(z+id)-g(z)$ is a nonconstant, then $g(z+md)-g(z)$ is a nonconstant for each $m=1,2,\cdots,s$, since a,b,b_m are non-zero, by applying Lemma 2.3 to (4.19) and (4.20) respectively, we get $ia_0\sqrt{b}+\sqrt{a}=ia_0\sqrt{b}-\sqrt{a}=0$, $b_mh_1(z+md)=0$, $b_mh_2(z+md)=0$ for all $m=1,2,\cdots,s$, this is a contradiction to our assumption that $a\neq 0$. Hence, there exists i $(i=1,\cdots,s)$, such that $g(z+id)-g(z)$ is a constant, then $g(z)$ is a polynomial.

Thus, it follows that $g(z)=L_2(z)+H_2(s)+B_2$, where $L_2(z),H_2(s),B_2$ are defined in Subcase 1.1.1, $p(z)$ and $g(z)$ satisfy the following relationship

$$\begin{cases} i\sqrt{b}\sum_{m=1}^{s} b_m h_2(z+md)e^{-L_1(md)}e^{\frac{L_2(md)}{2}}+(ia_0\sqrt{b}+\sqrt{a})=0,\\ i\sqrt{b}\sum_{m=1}^{s} b_m h_1(z+md)e^{L_1(md)}e^{\frac{L_2(md)}{2}}+(ia_0\sqrt{b}-\sqrt{a})=0. \end{cases}$$

Subcase 2.1.2: If

$$-i\sqrt{b}\sum_{m=1}^{s} b_m h_2(z+md)e^{-L_1(md)}e^{\frac{g(z+md)-g(z)}{2}}+(-ia_0\sqrt{b}-\sqrt{a})\neq 0.$$

Then, it follows from (4.18) that

$$\begin{aligned} &i\sqrt{b}\sum_{m=1}^{s} b_m h_1(z+md)e^{L_1(md)}e^{\frac{g(z+md)-g(z)}{2}}+(ia_0\sqrt{b}-\sqrt{a})\\ =&\left(-i\sqrt{b}\sum_{m=1}^{s} b_m h_2(z+md)e^{-L_1(md)}e^{\frac{g(z+md)-g(z)}{2}}+(-ia_0\sqrt{b}-\sqrt{a})\right)e^{-2p(z)}. \end{aligned} \tag{4.21}$$

If $p(z)$ is a nonconstant polynomial, from (4.21) and Lemma 2.3, this is a contradiction.

If $p(z)$ is a constant, then $L_1(md)=0$, $h_1(z)=h_2(z)$. It follows from (4.21) that

$$\begin{aligned} &\left(-ie^{-2p(z)}\sqrt{b}b_m h_2(z+md)-i\sqrt{b}b_m h_1(z+md)\right)e^{\frac{g(z+md)-g(z)}{2}}\\ &\quad+\left(e^{-2p(z)}(-ia_0\sqrt{b}-\sqrt{a})-(ia_0\sqrt{b}-\sqrt{a})\right)=0, \quad (m=1,\cdots,s), \end{aligned} \tag{4.22}$$

if there exists i $(i=1,\cdots,s)$, such that $g(z+id)-g(z)$ is a nonconstant, then $g(z+md)-g(z)$ is a nonconstant for each $m=1,2,\cdots,s$, by applying Lemma 2.3 to (4.22), we have

$$-ie^{-2p(z)}\sqrt{b}b_mh_2(z+md)-i\sqrt{b}b_mh_1(z+md)=0, \quad (m=1,\cdots,s),$$

and

$$e^{-2p(z)}(-ia_0\sqrt{b}-\sqrt{a})=ia_0\sqrt{b}-\sqrt{a},$$

since a, b, b_m $(m=1,\cdots,s)$ are non-zero, then from the first equation above, we have $(e^{-2p(z)}+1)=0$, substitute it into the second equation above, we obtain $(e^{-2p(z)}-1)=0$, this is a contradiction. Hence, there exists i $(i=1,\cdots,s)$, such that $g(z+id)-g(z)$ is a constant, then $g(z)$ is a polynomial.

Subcase 2.2: Assume that $p(z+c)-p(z)$ is nonconstant, then by similar arguments of Subcase 1.2, we can also get a contradiction.

Case 3: Suppose that $L_1^k(f)\not\equiv 0$ and $L_2^s(f)\not\equiv 0$ and $c=d\in\mathbb{C}^n$, then, by (4.3), (4.4) and (4.15), we can get a simple formula as follows

$$\begin{aligned}
&i\sqrt{b}\sum_{j=1}^{k}a_je^{p(z+jc)+p(z)}e^{\frac{g(z+jc)-g(z)}{2}}+i\sqrt{b}\sum_{j=1}^{k}a_je^{p(z)-p(z+jc)}e^{\frac{g(z+jc)-g(z)}{2}}\\
&+i\sqrt{b}\sum_{m=1}^{s}b_mh_1(z+mc)e^{p(z+mc)+p(z)}e^{\frac{g(z+mc)-g(z)}{2}}\\
&+i\sqrt{b}\sum_{m=1}^{s}b_mh_2(z+mc)e^{p(z)-p(z+mc)}e^{\frac{g(z+mc)-g(z)}{2}}\\
&+(ia_0\sqrt{b}-\sqrt{a})e^{2p(z)}=(-ia_0\sqrt{b}-\sqrt{a}).
\end{aligned} \tag{4.23}$$

Next we consider whether $p(z+c)-p(z)$ is constant.

Subcase 3.1: $p(z+c)-p(z)=\eta$, where η is a constant in $\mathbb{C}$. Since $p(z)$ is a polynomial in $\mathbb{C}^n$, then we have

$$p(z)=L_1(z)+H_1(s)+B_1,$$

where $L_1(z), H_1(s), B_1$ are defined in Subcase 1.1. Thus, we get $p(z+jc)-p(z)=L_1(jc)$, $p(z+mc)-p(z)=L_1(mc)$ for all $j,m\in\mathbb{N}$. It follows from (4.23) that

$$\begin{aligned}
&i\sqrt{b}\sum_{j=1}^{k}a_je^{p(z+jc)+p(z)}e^{\frac{g(z+jc)-g(z)}{2}}+i\sqrt{b}\sum_{m=1}^{s}b_mh_1(z+mc)e^{p(z+mc)+p(z)}e^{\frac{g(z+mc)-g(z)}{2}}\\
&\quad+(ia_0\sqrt{b}-\sqrt{a})e^{2p(z)}\\
=&-i\sqrt{b}\sum_{j=1}^{k}a_je^{-L_1(jc)}e^{\frac{g(z+jc)-g(z)}{2}}-i\sqrt{b}\sum_{m=1}^{s}b_mh_2(z+mc)e^{-L_1(mc)}e^{\frac{g(z+mc)-g(z)}{2}}\\
&\quad+(-ia_0\sqrt{b}-\sqrt{a}).
\end{aligned} \tag{4.24}$$

Subcase 3.1.1: If

$$
\begin{aligned}
&-i\sqrt{b}\sum_{j=1}^{k}a_je^{-L_1(jc)}e^{\frac{g(z+jc)-g(z)}{2}}-i\sqrt{b}\sum_{m=1}^{s}b_mh_2(z+mc)e^{-L_1(mc)}e^{\frac{g(z+mc)-g(z)}{2}}\\
&\quad+(-ia_0\sqrt{b}-\sqrt{a})=0,
\end{aligned}
\tag{4.25}
$$

then, it follows from (4.24) that

$$
\begin{aligned}
&i\sqrt{b}\sum_{j=1}^{k}a_je^{L_1(jc)}e^{\frac{g(z+jc)-g(z)}{2}}+i\sqrt{b}\sum_{m=1}^{s}b_mh_1(z+mc)e^{L_1(mc)}e^{\frac{g(z+mc)-g(z)}{2}}\\
&\quad+(ia_0\sqrt{b}-\sqrt{a})=0.
\end{aligned}
\tag{4.26}
$$

If there exists i ($i\in\mathbb{N}$), such that $g(z+ic)-g(z)$ is nonconstant, then $g(z+jc)-g(z)$ and $g(z+mc)-g(z)$ are nonconstant for each $j,m\in\mathbb{N}$, since a,b are non-zero and at least one of a_j or b_m are non-zero, by applying Lemma 2.3 to (4.25) and (4.26) respectively, we get $ia_0\sqrt{b}+\sqrt{a}=ia_0\sqrt{b}-\sqrt{a}=0$, $a_j=0$, $b_mh_1(z+mc)=0$, $b_mh_2(z+mc)=0$ for all $j=1,2,\cdots,k$, $m=1,2,\cdots,s$, this is a contradiction to our assumption that $a\neq 0$. Hence, there exists i ($i\in\mathbb{N}$), such that $g(z+ic)-g(z)$ is a constant, then $g(z)$ is a polynomial.

Thus, it follows that $g(z)=L_2(z)+H_2(s)+B_2$, where $L_2(z)$, $H_2(s)$, B_2 are defined in Subcase 1.1.1, $p(z)$ and $g(z)$ satisfy the following relationship

$$
\begin{cases}
i\sqrt{b}\displaystyle\sum_{j=1}^{k}a_je^{-L_1(jc)}e^{\frac{L_2(jc)}{2}}+i\sqrt{b}\sum_{m=1}^{s}b_mh_2(z+mc)e^{-L_1(mc)}e^{\frac{L_2(mc)}{2}}+(ia_0\sqrt{b}+\sqrt{a})=0,\\
i\sqrt{b}\displaystyle\sum_{j=1}^{k}a_je^{L_1(jc)}e^{\frac{L_2(jc)}{2}}+i\sqrt{b}\sum_{m=1}^{s}b_mh_1(z+mc)e^{L_1(mc)}e^{\frac{L_2(mc)}{2}}+(ia_0\sqrt{b}-\sqrt{a})=0.
\end{cases}
$$

Subcase 3.1.2: If

$$
\begin{aligned}
&-i\sqrt{b}\sum_{j=1}^{k}a_je^{-L_1(jc)}e^{\frac{g(z+jc)-g(z)}{2}}-i\sqrt{b}\sum_{m=1}^{s}b_mh_2(z+mc)e^{-L_1(mc)}e^{\frac{g(z+mc)-g(z)}{2}}\\
&\quad+(-ia_0\sqrt{b}-\sqrt{a})\neq 0,
\end{aligned}
$$

then, it follows from (4.24) that

$$
\begin{aligned}
&i\sqrt{b}\sum_{j=1}^{k}a_je^{L_1(jc)}e^{g_1(z)}+i\sqrt{b}\sum_{m=1}^{s}b_mh_1(z+mc)e^{L_1(mc)}e^{g_2(z)}+(ia_0\sqrt{b}-\sqrt{a})\\
=&\left(-i\sqrt{b}\sum_{j=1}^{k}a_je^{-L_1(jc)}e^{g_1(z)}-i\sqrt{b}\sum_{m=1}^{s}b_mh_2(z+mc)e^{-L_1(mc)}e^{g_2(z)}\right.\\
&\quad\left.+(-ia_0\sqrt{b}-\sqrt{a})\right)e^{-2p(z)},
\end{aligned}
\tag{4.27}
$$

where

$$g_1(z)=\frac{g(z+jc)-g(z)}{2}\quad\text{and}\quad g_2(z)=\frac{g(z+mc)-g(z)}{2}.$$

If $p(z)$ is a nonconstant polynomial, from (4.27) and Lemma 2.3, this is a contradiction.

If $p(z)$ is a constant, then $L_1(mc)=L_1(jc)=0$, $h_1(z)=h_2(z)$.

When $k=s$, it follows from (4.27) that

$$\begin{aligned}&\Big(-e^{-2p(z)}i\sqrt{b}a_j-i\sqrt{b}a_j-e^{-2p(z)}i\sqrt{b}b_jh_2(z+jc)-i\sqrt{b}b_jh_1(z+jc)\Big)e^{\frac{g(z+jc)-g(z)}{2}}\\&\quad+\Big(e^{-2p(z)}(-ia_0\sqrt{b}-\sqrt{a})-(ia_0\sqrt{b}-\sqrt{a})\Big)=0,\quad (j=1,\cdots,k),\end{aligned}\tag{4.28}$$

if there exists i $(i=1,\cdots,k)$, such that $g(z+ic)-g(z)$ is a nonconstant, then $g(z+jc)-g(z)$ is a nonconstant for each $j=1,2,\cdots,k$, by applying Lemma 2.3 to (4.28)

$$\begin{aligned}&-e^{-2p(z)}i\sqrt{b}a_j-i\sqrt{b}a_j-e^{-2p(z)}i\sqrt{b}b_jh_2(z+jc)\\&\quad-i\sqrt{b}b_jh_1(z+jc)=0,\quad (j=1,\cdots,k),\end{aligned}\tag{4.29}$$

and

$$e^{-2p(z)}(-ia_0\sqrt{b}-\sqrt{a})=ia_0\sqrt{b}-\sqrt{a},\tag{4.30}$$

(4.29) can be written as

$$-e^{-2p(z)}i\sqrt{b}[a_j+b_jh_2(z+jc)]=i\sqrt{b}[a_j+b_jh_1(z+jc)],\quad (j=1,\cdots,k),$$

since a, b are non-zero and at least one of a_j of b_m are non-zero, then we have $(e^{-2p(z)}+1)=0$, substitute it into (4.30), we obtain $(e^{-2p(z)}-1)=0$, this is a contradiction. Hence, there exists i $(i=1,\cdots,k)$, such that $g(z+ic)-g(z)$ is a constant, then $g(z)$ is a polynomial.

When $k\neq s$, then $k>s$ or $k<s$.

When $k>s$, it follows from (4.27) that

$$\begin{aligned}&\Bigg(-e^{-2p(z)}i\sqrt{b}\sum_{j=1}^{s}a_j-i\sqrt{b}\sum_{j=1}^{s}a_j-e^{-2p(z)}i\sqrt{b}\sum_{j=1}^{s}b_jh_2(z+jc)\\&\quad-i\sqrt{b}\sum_{j=1}^{s}b_jh_1(z+jc)\Bigg)e^{\frac{g(z+jc)-g(z)}{2}}+\Bigg(-ie^{-2p(z)}\sqrt{b}\sum_{j=s+1}^{k}a_j-i\sqrt{b}\sum_{j=s+1}^{k}a_j\Bigg)e^{\frac{g(z+jc)-g(z)}{2}}\\&\quad+\Big(e^{-2p(z)}(-ia_0\sqrt{b}-\sqrt{a})-(ia_0\sqrt{b}-\sqrt{a})\Big)=0,\end{aligned}\tag{4.31}$$

if there exists i $(i=1,\cdots,k)$, such that $g(z+ic)-g(z)$ is a nonconstant, then $g(z+jc)-g(z)$ is a nonconstant for each $j=1,2,\cdots,k$, by applying Lemma 2.3 to (4.33), we have

$$\begin{aligned}&-e^{-2p(z)}i\sqrt{b}a_j-i\sqrt{b}a_j-e^{-2p(z)}i\sqrt{b}b_jh_2(z+jc)\\&\quad-i\sqrt{b}b_jh_1(z+jc)=0,\quad (j=1,\cdots,s),\end{aligned}\tag{4.32}$$

and

$$-ie^{-2p(z)}\sqrt{b}a_j - i\sqrt{b}a_j = 0, \quad (j = s+1, \cdots, k), \tag{4.33}$$

and

$$e^{-2p(z)}(-ia_0\sqrt{b} - \sqrt{a}) = ia_0\sqrt{b} - \sqrt{a}, \tag{4.34}$$

(4.32) can be written as

$$-e^{-2p(z)}i\sqrt{b}[a_j + b_j h_2(z+jc)] = i\sqrt{b}[a_j + b_j h_1(z+jc)], \quad (j = 1, \cdots, s), \tag{4.35}$$

since a, b are non-zero and at least one of a_j of b_m are non-zero, then from (4.33) and (4.35), we all have $(e^{-2p(z)} + 1) = 0$, substitute it into (4.34), we obtain $(e^{-2p(z)} - 1) = 0$, this is a contradiction. Hence, there exists i $(i = 1, \cdots, k)$, such that $g(z+ic) - g(z)$ is a constant, then $g(z)$ is a polynomial.

Similarly, when $k < s$, we can get a contradiction.

Subcase 3.2: Suppose that $p(z+c) - p(z)$ is nonconstant. Hence, we can easily get $p(z+jc) - p(z)$ is nonconstant for each $j = 1, 2, \cdots, k$ and $p(z+mc) - p(z)$ is nonconstant for each $j = 1, 2, \cdots, s$. By applying Lemma 2.3 to (4.23), we discover that

$$\begin{aligned} &ia_0\sqrt{b} + \sqrt{a} = ia_0\sqrt{b} - \sqrt{a} = 0, \quad b_m h_1(z+mc) = 0, \quad b_m h_2(z+mc) = 0 \\ &\quad \text{and} \quad a_j = 0 \quad \text{for all } j = 1, 2, \cdots, k. \end{aligned}$$

This is a contradiction to our assumption that $a \neq 0$. This completes the proof of Theorem 4.1. □

Acknowledgements

This work was supported by National Natural Science Foundation of China (Grant Nos. 12261023, 11861023), Doctoral Research Project funded by Guizhou Normal University (No. GZNUD [2019] No. 16) and Guizhou Normal University Academic New Seedlings Fund Project Class B (Qianshi New Seedlings [2022] No. B17) funded project.

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