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The Character of Thurston's Circle Packings with Obtuse Exterior Intersection Angles

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Abstract. We study the character of Thurston's circle packings with obtuse exterior intersection angles, and we get some simple criteria for the existence of hyperbolic circle packings. Moreover, the compactness theorem of combinatorial Ricci flows in hyperbolic background geometry with the weight function $\Phi \in [0, \pi)$ is obtained. As a consequence, We generalize G. Lin's result [12] from acute exterior intersection angles case to obtuse exterior intersection angles case.

Key Words: Character, Thurston's circle packing, combinatorial Ricci flow, obtuse exterior intersection angles.

AMS Subject Classifications: 52C26, 53A70, 53E99, 57Q15

1 Introduction

Circle packings were first studied by P. Koebe [15] in the 1930s in the context of conformal mapping, but the topic quickly dropped from sight. In the 1970s, W. Thurston [20] discovered circle packings independently in the process of constructing certain hyperbolic 3-manifolds. In 1985, W. Thurston [21] recognized some special character of rigidity in these circle configurations that was reminiscent of that shown by anlytic functions, so he conjectured that people could use circle packings to approximate classical conformal mapping. In 1987, R. Sullivan [18] proved Thurston's conjecture.

In fact, in the 1970s, W. Thurston [20] observed a very deep connnection between circle packings and hyperbolic polyhedra. Given a convex hyperbolic polyhedra in the hyperbolic 3-space $\mathbb{B}^3$, the boundaries of the oriented hyperbolic planes containing its faces form a circle packing on the sphere $\partial\mathbb{B}^3$. This circle packings record all the information of the original polyhedron.

W.Thurston's hyperbolization theorem for 3-manifolds is an important discovery in mathematics, it builds a connection between the geometry and topology of 3-manifolds

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and the algebra of discrete groups of $Isom(\mathbb{H}^3)$. In 1970, E. Andreev [1,2] provided a complete characterization of compact hyperbolic polyhedra with non-obtuse dihedral angles, and this theorem is one of the three main tools in proving Thurston's hyperbolization theorem for Haken 3-manifolds. Combining the work of P. Koebe [15], E. Andreev [1,2] and W. Thurston [20], which is usually called the Koebe-Andreev-Thurston theorem, people obtained a complete criterion for the existence of circle packings.

From Ge-Lin's point of view [12], the criteria of Koebe-Andreev-Thurston theorem are difficult to verify, so Ge-Lin introduced the character of circle packings, and they obtained some simple criteria for the existence of circle packings for non-obtuse exterior intersection angles.

In this paper, we study the character of Thurston's circle packings with obtuse exterior intersection angles $\Phi \in [0,\pi)$, and we also get some simple criteria for the existence of hyperbolic circle packings.

Suppose X is a closed surface and $\mathcal{T}$ is a triangulation on X. Let $V=\{v_1,\cdots,v_n\}$ be the set of vertices in $\mathcal{T}$, where n is the number of vertices. Let e_{ij} be the edge joining v_i and v_j, we denote the set of all edges and triangles in $\mathcal{T}$ by E and F. The weight function on the triangulation is $\Phi: E\to[0,\pi)$.

A **circle packing** $\mathcal{P}=\{C_v : v\in V\}$ on a surface is a collection of circles with a particular combinatorial structure. Suppose X is equipped with a constant curvature metric μ, we say a circle packing $\mathcal{P}$ on (X,μ) is called (T,Φ)-type if there exists a geodesic triangulation $\mathcal{T}_\mu$ on (X,μ) isotopic to $\mathcal{T}$ such that the circle C_v is centered at $\mathcal{T}_\mu(v)$ and for any edge $e\in E$, the two circles C_u, C_v which correspond to the vertices u, v of e intersecting at an angle $\Phi(e)$.

Given a weight function $\Phi: E\to[0,\pi)$, a natural question is:

Question 1.1. Does there exists a $(\mathcal{T},\Phi)$-type circle packing $\mathcal{P}$ on (X,μ) whose exterior intersection angle function is given by Φ? If it does, to what extent is the circle packing unique?

Suppose $(X,\mathcal{T},\Phi)$ is a triangulated closed surface with weight function $\Phi: E\to[0,\pi)$, for any circle packings based on $(X,\mathcal{T},\Phi)$, in order to give some criteria for the existence of hyperbolic circle packings, Ge-Lin [12] introduced the character of the weighted triangulation surfaces $(X,\mathcal{T},\Phi)$ as

$$\mathcal{L}(\mathcal{T},\Phi)=(\mathcal{L}(\mathcal{T},\Phi)_1,\cdots,\mathcal{L}(\mathcal{T},\Phi)_n),$$

where $\mathcal{L}(\mathcal{T},\Phi)_i$ is the character at each vertex $i\in V$,

$$\mathcal{L}(\mathcal{T},\Phi)_i=\sum_{\triangle ijk\in F}\arccos\Big(\frac{1+\cos\Phi_{ij}+\cos\Phi_{ki}-\cos\Phi_{jk}}{2\sqrt{1+\cos\Phi_{ij}}\sqrt{1+\cos\Phi_{ki}}}\Big). \tag{1.1}$$

For a three-circle configuration, as shown in Fig. 1, Zhou [22] introduced the notation λ_{ijk} as

$$\lambda_{ijk}=\cos\Phi_{ij}+\cos\Phi_{jk}\cos\Phi_{ki},$$