Analysis in Theory and Applications
Anal. Theory Appl., Vol. **41**, No. 1 (2025), pp. 80-98

DOI: 10.4208/ata.OA-2024-0046
March 2025

The Character of Thurston's Circle Packings with Obtuse Exterior Intersection Angles

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Received 12 November 2024; Accepted (in revised version) 6 March 2025

Abstract. We study the character of Thurston's circle packings with obtuse exterior intersection angles, and we get some simple criteria for the existence of hyperbolic circle packings. Moreover, the compactness theorem of combinatorial Ricci flows in hyperbolic background geometry with the weight function $\Phi \in [0, \pi)$ is obtained. As a consequence, We generalize G. Lin's result [12] from acute exterior intersection angles case to obtuse exterior intersection angles case.

Key Words: Character, Thurston's circle packing, combinatorial Ricci flow, obtuse exterior intersection angles.

AMS Subject Classifications: 52C26, 53A70, 53E99, 57Q15

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1 Introduction

Circle packings were first studied by P. Koebe [15] in the 1930s in the context of conformal mapping, but the topic quickly dropped from sight. In the 1970s, W. Thurston [20] discovered circle packings independently in the process of constructing certain hyperbolic 3-manifolds. In 1985, W. Thurston [21] recognized some special character of rigidity in these circle configurations that was reminiscent of that shown by anlytic functions, so he conjectured that people could use circle packings to approximate classical conformal mapping. In 1987, R. Sullivan [18] proved Thurston's conjecture.

In fact, in the 1970s, W. Thurston [20] observed a very deep connnection between circle packings and hyperbolic polyhedra. Given a convex hyperbolic polyhedra in the hyperbolic 3-space $\mathbb{B}^3$, the boundaries of the oriented hyperbolic planes containing its faces form a circle packing on the sphere $\partial\mathbb{B}^3$. This circle packings record all the information of the original polyhedron.

W.Thurston's hyperbolization theorem for 3-manifolds is an important discovery in mathematics, it builds a connection between the geometry and topology of 3-manifolds

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and the algebra of discrete groups of $Isom(\mathbb{H}^3)$. In 1970, E. Andreev [1,2] provided a complete characterization of compact hyperbolic polyhedra with non-obtuse dihedral angles, and this theorem is one of the three main tools in proving Thurston's hyperbolization theorem for Haken 3-manifolds. Combining the work of P. Koebe [15], E. Andreev [1,2] and W. Thurston [20], which is usually called the Koebe-Andreev-Thurston theorem, people obtained a complete criterion for the existence of circle packings.

From Ge-Lin's point of view [12], the criteria of Koebe-Andreev-Thurston theorem are difficult to verify, so Ge-Lin introduced the character of circle packings, and they obtained some simple criteria for the existence of circle packings for non-obtuse exterior intersection angles.

In this paper, we study the character of Thurston's circle packings with obtuse exterior intersection angles $\Phi \in [0,\pi)$, and we also get some simple criteria for the existence of hyperbolic circle packings.

Suppose X is a closed surface and $\mathcal{T}$ is a triangulation on X. Let $V=\{v_1,\cdots,v_n\}$ be the set of vertices in $\mathcal{T}$, where n is the number of vertices. Let e_{ij} be the edge joining v_i and v_j, we denote the set of all edges and triangles in $\mathcal{T}$ by E and F. The weight function on the triangulation is $\Phi: E\to[0,\pi)$.

A **circle packing** $\mathcal{P}=\{C_v : v\in V\}$ on a surface is a collection of circles with a particular combinatorial structure. Suppose X is equipped with a constant curvature metric μ, we say a circle packing $\mathcal{P}$ on (X,μ) is called (T,Φ)-type if there exists a geodesic triangulation $\mathcal{T}_\mu$ on (X,μ) isotopic to $\mathcal{T}$ such that the circle C_v is centered at $\mathcal{T}_\mu(v)$ and for any edge $e\in E$, the two circles C_u, C_v which correspond to the vertices u, v of e intersecting at an angle $\Phi(e)$.

Given a weight function $\Phi: E\to[0,\pi)$, a natural question is:

Question 1.1. Does there exists a $(\mathcal{T},\Phi)$-type circle packing $\mathcal{P}$ on (X,μ) whose exterior intersection angle function is given by Φ? If it does, to what extent is the circle packing unique?

Suppose $(X,\mathcal{T},\Phi)$ is a triangulated closed surface with weight function $\Phi: E\to[0,\pi)$, for any circle packings based on $(X,\mathcal{T},\Phi)$, in order to give some criteria for the existence of hyperbolic circle packings, Ge-Lin [12] introduced the character of the weighted triangulation surfaces $(X,\mathcal{T},\Phi)$ as

$$\mathcal{L}(\mathcal{T},\Phi)=(\mathcal{L}(\mathcal{T},\Phi)_1,\cdots,\mathcal{L}(\mathcal{T},\Phi)_n),$$

where $\mathcal{L}(\mathcal{T},\Phi)_i$ is the character at each vertex $i\in V$,

$$\mathcal{L}(\mathcal{T},\Phi)_i=\sum_{\triangle ijk\in F}\arccos\Big(\frac{1+\cos\Phi_{ij}+\cos\Phi_{ki}-\cos\Phi_{jk}}{2\sqrt{1+\cos\Phi_{ij}}\sqrt{1+\cos\Phi_{ki}}}\Big). \tag{1.1}$$

For a three-circle configuration, as shown in Fig. 1, Zhou [22] introduced the notation λ_{ijk} as

$$\lambda_{ijk}=\cos\Phi_{ij}+\cos\Phi_{jk}\cos\Phi_{ki},$$

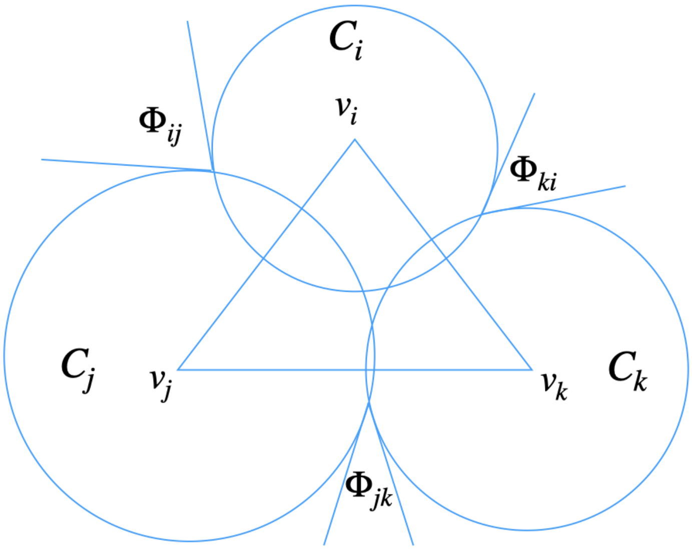

Figure 1: A three-circle configuration.

similarly, they denoted

$$\lambda_{jki} = \cos\Phi_{jk} + \cos\Phi_{ki}\cos\Phi_{ij},$$
$$\lambda_{kij} = \cos\Phi_{ki} + \cos\Phi_{ij}\cos\Phi_{jk}.$$

Now, we state our main results. We study the character of Thurston's circle packing when the weight function is $\Phi : E \to [0,\pi)$, In this section, we always assume that $\lambda_{ijk}, \lambda_{jki}, \lambda_{kij} \geq 0$ for all $\triangle ijk \in F$.

Theorem 1.1. *Let X be a closed surface. If X admits a triangulation $\mathcal{T}$ with degree $d \geq 7$ at each vertex, assume that $\lambda_{ijk}, \lambda_{jki}, \lambda_{kij} \geq 0$ for all $\triangle ijk \in F$, then for any given constant weight $\Phi : E \to [0,\pi)$, there exists a unique complete hyperbolic metric μ on X so that (X,μ) supports a $(\mathcal{T},\Phi)$-type circle packing $\mathcal{P}$.*

Theorem 1.1 is a simple criteria for the existence of hyperbolic circle packing. Since Beardon-Stephenson [3] have pointed out that every circle packing $\mathcal{P}$ with all vertex degree $d \leq 6$ can not be supported on any closed surface X of genus $g > 2$, and every circle packing $\mathcal{P}$ with all vertex degree $d \geq 7$ can not be supported on the sphere or torus, so the requirement $d \geq 7$ is sharp.

However, the condition that the weight function Φ is a constant in Theorem 1.1 is a little bit restrictive, and it can be released to some extent. In fact, we have the following generalization.

Theorem 1.2. *Let X be a closed surface. If X admits a triangulation $\mathcal{T}$ with degree $d \geq 7$ at each vertex, assume that $\lambda_{ijk}, \lambda_{jki}, \lambda_{kij} \geq 0$ for all $\triangle ijk \in F$, then for any weight $\Phi \in [\arccos \eta, \arccos \xi] \subset [0, \pi/2]$, where $0 \leq \xi \leq \eta \leq 1$, and for any $\Phi \in [\arccos \eta, \arccos \xi] \subset [\pi/2, \pi)$ where $-1 < \xi \leq \eta \leq 0$, η and ξ are arbitrary chosen so that $\eta < (2\cos\frac{2\pi}{7} - 1 + \xi)/(2 - 2\cos\frac{2\pi}{7})$, there exists a unique complete hyperbolic metric μ on X, so that (X, μ) supports a $(\mathcal{T}, \Phi)$-type circle packing $\mathcal{P}$.*

For instance, if Φ takes values in $[0, 0.33\pi]$ or in $[0.4\pi, \pi/2]$ respectively, they all satisfy the assumption in Theorem 1.2.

Theorem 1.3. *Given a weighted triangulated closed surface $(X, \mathcal{T}, \Phi)$ with the weight $\Phi : E \to [0, \pi)$, assume that $\lambda_{ijk}, \lambda_{jki}, \lambda_{kij} \geq 0$ for all $\triangle ijk \in F$, consider the hyperbolic background setting.*

(a) If the character $\mathcal{L}(\mathcal{T}, \Phi)_i > 2\pi$ at all vertices, then $\chi(X) < 0$, and there exists a unique hyperbolic circle packing r_{ze} based on $(X, \mathcal{T}, \Phi)$. Consequently, r_{ze} determines a unique complete hyperbolic metric μ on X, such that (X, μ) supports a geometric decomposition isotopic to $\mathcal{T}$, and each edge connecting two adjacent vertices is a hyperbolic geodesic.

(b) If the character $\mathcal{L}(\mathcal{T}, \Phi)_i \leq 2\pi$ at all vertices, then $\chi(X) \geq 0$. Consequently, there exists no hyperbolic circle packings based on $(X, \mathcal{T}, \Phi)$. In this case, any solution $r(t)$ to flow (2.1) *satifies $r(t) \to 0$ when $t \to \infty$.*

Theorem 1.4. *Given a weighted triangulated closed surface $(X, \mathcal{T}, \Phi)$ with the weight $\Phi : E \to [0, \pi)$, assume that $\lambda_{ijk}, \lambda_{jki}, \lambda_{kij} \geq 0$ for all $\triangle ijk \in F$,*

(a) If the character $\mathcal{L}(\mathcal{T}, \Phi)_i > 2\pi$ at all vertices, then $\chi(X) < 0$ and there exists a unique complete hyperbolic metric μ on X, such that (X, μ) supports a $(\mathcal{T}, \Phi)$-type circle packing $\mathcal{P}$.

(b) On the other hand, if the character $\mathcal{L}(\mathcal{T}, \Phi)_i \leq 2\pi$ at all vertices, then $\chi(X) \geq 0$ and there exists no such $(\mathcal{T}, \Phi)$-type circle packings.

In our paper, one of the main tools is the convergence of the solution to combinatorial Ricci flow in hyperbolic background geometry. In Chow-Luo's paper [6], they request $\Phi \in [0, \pi/2]$ and they get the convergence result of combinatorial Ricci flow in hyperbolic background geometry, we generalize the condition to $\Phi \in [0, \pi)$ and have the following compactness theorem.

Lemma 1.1. *Given a weighted triangulated closed surface $(X, \mathcal{T}, \Phi)$ with $\Phi : E \to [0, \pi)$, assume that $\lambda_{ijk}, \lambda_{jki}, \lambda_{kij} \geq 0$ for all $\triangle ijk \in F$. Suppose $r(t)$ for $t \in [0, +\infty)$ is a solution to the Ricci flow (2.1) in hyperbolic background geometry so that the set $\{r(t) | t \in (0, +\infty)\}$ lies in a compact region in $\mathbb{R}^N_{>0}$, then $r(t)$ converges exponentially fast to a circle packing metric in $\mathbb{R}^N_{>0}$ whose curvature at each vertex is zero.*

This paper is organized as follows. In Section 1, we introduce the background and main results. In Section 2, we introduce some preliminaries. In Section 3, we introduce the three-circle configuration and the character of circle packings. In Section 4, we introduce some basic knowledge of combinatorial Ricci flow on surfacess. In Section 5, we prove the main results.

2 Preliminaries

In our case, we consider the hyperbolic circle packings. Given a triangulated closed surface $(X,\mathcal{T},\Phi)$ with weight $\Phi: E\to[0,\pi)$, we assign each vertex v_i a positive number r_i, and realize each edge e_{ij} by a hyperbolic segment of length

$$l_{ij}=\cosh^{-1}(\cosh r_i\cosh r_j+\sinh r_i\sinh r_j\cos\Phi_{ij}).$$

Hence, we can realize each triangle $\triangle ijk\in F$ by a hyperbolic triangle of edge length l_{ij}, l_{jk}, l_{ki}.

The triangle is formed by the centers of three circles of radii r_i, r_j and r_k intersecting at angles $\Phi(e_{ij})$, $\Phi(e_{jk})$ and $\Phi(e_{ki})$, for the simplify of notation, we denote $\Phi(e_{ij})$, $\Phi(e_{jk})$ and $\Phi(e_{ki})$ by Φ_{ij}, Φ_{jk} and Φ_{ki}, respectively. The inner angle at v_i, v_j and v_k in triangle $\triangle ijk$ is θ_i, θ_j and θ_k, respectively. This produces a hyperbolic cone metric on the surface X with singularities at the vertices. We call these radii $(r_1,\cdots,r_n)$ the circle-packing metrics based on $(\mathcal{T},\Phi)$.

Let a_i be the cone angle at the vertex v_i, where cone angle means the sum of all inner angles having vertex v_i. The discrete Gauss curvature K_i at each vertex v_i is defined to be $2\pi-a_i$. Given a circle packing metric $r=(r_1,\cdots,r_n)$, the corresponding curvature is $K=(K_1,\cdots,K_n)$.

Consider the curvature map

$$\begin{aligned} K:\ & r\mapsto K,\ r\in\mathbb{R}^n_{>0},\\ & (r_1,\cdots,r_n)\mapsto(K_1,\cdots,K_n). \end{aligned}$$

The Koebe-Andreev-Thurston theorem [17] says that the curvature map $K: r\mapsto K$ is injective and its image set $K(\mathbb{R}^n_{>0})$ is a bounded convex polytope.

Let X be a closed surface with a triangulation $\mathcal{T}=(V,E,F)$, where V, E, F represents the set of vertices, edges and triangles, respectively. Given a weight function $\Phi: E\to[0,\pi)$, suppose X is equipped with a constant curvature metric μ.

In the non-obtuse intersection angle case, that is, for non-obtuse weight function $\Phi: E\to[0,\pi/2]$, Thurston's circle packing theorem [20] has answered the existence and uniqueness question of $(\mathcal{T},\Phi)-type$ circle packing on X. More precisely, W.Thurston studyed the existence and rigidity of circle packing on higher genus surfaces with a prescribed combinatorial type and non-obtuse exterior intersection angles.

Theorem 2.1 (W. Thurston, [20]). *Let $\mathcal{T}$ be a triangulation of a closed surface X of genus $g>0$. Let $\Phi: E\to[0,\pi/2]$ be a function satisfying the following conditions:*

(C1) *If e_1, e_2, e_3 form a null-homotopic closed path in $\mathcal{T}$, and $\sum_{l=1}^{3}\Phi(e_l)\geq\pi$, then these edges form the boundary of a triangle of $\mathcal{T}$;*

(C2) *If e_1, e_2, e_3, e_4 form a null-homotopic closed path and $\sum_{l=1}^{4}\Phi(e_l)\geq 2\pi$, then e_1, e_2, e_3, e_4 form the boundary of the union of two adjacent triangles.*

Then there exists a constant curvature metric μ on X such that (X,μ) supports a (T,Φ)-type circle packing $\mathcal{P}$. Moreover, the pair $(\mathcal{P},\mu)$ is unique up to isometries if $g>1$ and up to similarity if $g=1$.

Thurston's circle packing theorem gives a complete criterion for the existence of circle packings, but the condition (C1) and (C2) are difficult to verify for general weight $\Phi\in[0,\pi/2]$, since the angle structure Φ and the combinatorial structure of $\mathcal{T}$ are globally intertwined together on X. Therefore, Ge-Lin [12] introduced the "character-type" criteria which consider the degree of the triangulation, their degree conditions do not respect the angle structure Φ, and they obtained some quite simple criteria for the existence of hyperbolic circle packings. For example, they concluded that if a closed surface X admits a circle packing with all vertex degree $d_i>7$, then it admits a unique complete hyperbolic metric, such that the triangulation graph of the circle packings is isotopic to a geometric decomposition of X.

For more related results on circle packing, we refer to the work of Colin de Verdière [7], Feng Luo [16], Marden-Rodin [17], Rodin-Sullivan [18], K. Stephenson [19] and others. In this paper, we generalize Ge-Lin's result to obtuse exterior intersection angles case.

In our proof, one of the main tools is the convergence of the solution to combinatorial Ricci flow on surfaces in hyperbolic background geometry, and we establish the compactness theorem of the solution to the flow.

Chow-Luo [6] introduced the following combinatorial Ricci flow on surfaces in hyperbolic background geometry

$$\frac{\mathrm{d}r_i}{\mathrm{d}t}=-K_i\sinh r_i, \tag{2.1}$$

where r_i is the circle packing metric and K_i is the discrete Gauss curvature. The combinatorial Ricci flow is the analogue of Hamilton's smooth Ricci flow in the discrete setting. The idea of combinatorial Ricci flow is similar to smooth Ricci flow, that is, to evolute the metric with the rate which is proportional to the value of corresponding curvature, and we want to see whether it will become constant everywhere.

As for the non-obtuse exterior intersection angle case, that is, when the weight function $\Phi: E\to[0,\pi/2]$, Chow-Luo [6] proved that the solution to combinatorial Ricci flow on surfaces in hyperbolic background geometry converges if and only if there exists zero curvature circle packing metrics.

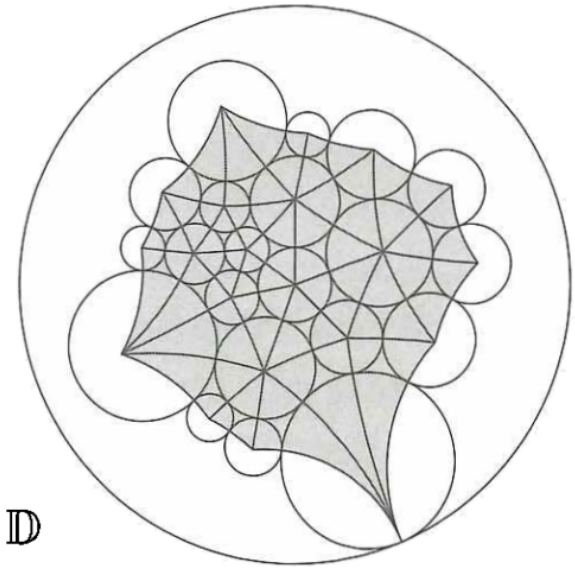

Figure 2: A tangential circle packing in hyperbolic background geometry [19, Fig. 1.6].

3 Three-circle configurations and the character of Thurston's circle packings

Since we want to generalize the weight function from acute exterior intersection angles case to obtuse exterior intersection angle case, we first show some results of three-circle configurations. The first lemma shows the existence and uniqueness of such a three-circle configuration.

Lemma 3.1 (Z. Zhou, [22]). *Suppose $\Phi_{ij}, \Phi_{jk}, \Phi_{ki} \in [0,\pi)$ satisfies*

$$\Phi_{ij}+\Phi_{jk}+\Phi_{ki} \leq \pi$$

or

$$\Phi_{ij}+\Phi_{jk} < \pi+\Phi_{ki}, \quad \Phi_{jk}+\Phi_{ki} < \pi+\Phi_{ij}, \quad \Phi_{ki}+\Phi_{ij} < \pi+\Phi_{jk}.$$

Then for any three positive numbers r_i, r_j, r_k, there exists a configuration of three mutually intersecting circles in hyperbolic background geometry, unique up to isometry, having radii r_i, r_j, r_k and meeting in exterior intersection angles Φ_{ij}, Φ_{jk}, Φ_{ki}.

In hyperbolic background geometry, for a given weight function $\Phi \in [0,\pi)$, in the proof of the convergence of combinatorial Ricci flows, one of the key observation is the following lemma. Recall that

$$\lambda_{ijk} = \cos\Phi_{ij} + \cos\Phi_{jk}\cos\Phi_{ki}.$$

Lemma 3.2 (Z. Zhou, [22]). *Suppose Φ_{ij}, Φ_{jk}, $\Phi_{ki} \in [0,\pi)$ satisfying $\lambda_{ijk} \geq 0$, $\lambda_{jki} \geq 0$, $\lambda_{kij} \geq 0$, then*

$$\frac{\partial \theta_i}{\partial r_i} < 0, \quad \sinh r_j \frac{\partial \theta_i}{\partial r_j} = \sinh r_i \frac{\partial \theta_j}{\partial r_i} \geq 0, \quad \frac{\partial(\theta_i + \theta_j + \theta_k)}{\partial r_i} < 0.$$

When $r_j > 0$, we have $\sinh r_j > 0$, hence $\frac{\partial \theta_i}{\partial r_j} \geq 0$ for $i \neq j$, this property plays an important role in our proof. But a natural question is: when can we achieve $\lambda_{ijk} \geq 0$, $\lambda_{jki} \geq 0$ and $\lambda_{kij} \geq 0$? The following lemma answers this question.

Lemma 3.3 (Z. Zhou, [22]). *Given Φ_{ij}, Φ_{jk}, $\Phi_{ki} \in [0,\pi)$, we have $\lambda_{ijk} \geq 0$, $\lambda_{jki} \geq 0$, $\lambda_{kij} \geq 0$ if and only if one of the following properties holds:*

1. $\Phi_{ij} + \Phi_{jk} + \Phi_{ki} \leq \pi$;
2. $\Phi_{ij}, \Phi_{jk}, \Phi_{ki}$ *are the angles of a spherical triangle with each side less than or equal to $\pi/2$.*

In Ge-Lin's paper [12], they defined the character of the weighted triangulation $(\mathcal{T}, \Phi)$ on X to be

$$\mathcal{L}(\mathcal{T}, \Phi) = (\mathcal{L}(\mathcal{T}, \Phi)_1, \cdots, \mathcal{L}(\mathcal{T}, \Phi)_n),$$

where

$$\mathcal{L}(\mathcal{T}, \Phi)_i = \sum_{\triangle ijk \in F} \arccos\Big(\frac{1 + \cos \Phi_{ij} + \cos \Phi_{ki} - \cos \Phi_{jk}}{2\sqrt{1 + \cos \Phi_{ij}}\sqrt{1 + \cos \Phi_{ki}}} \Big), \tag{3.1}$$

and they use the character to give a new criteria of the existence of circle packings.

Since the combinatorial structure $\mathcal{T}$ and the angle structure Φ come from the circle packings $\mathcal{P}$ on X, so the weighted triangulation $(\mathcal{T}, \Phi)$ records all the information of a $(\mathcal{T}, \Phi)$-type circle packings $\mathcal{P}$. Hence, the character $\mathcal{L}(\mathcal{T}, \Phi)_{i \in V}$ is indeed an invariant of all $(\mathcal{T}, \Phi)$-type circle packings $\mathcal{P}$ on a closed surface X.

Specially, Ge-Lin [12] proved that in Euclidean background geometry, for a given weighted triangulated surface $(X, \mathcal{T}, \Phi)$, if the circle packings metric $r = (1, \cdots, 1)$, then the character $\mathcal{L}(\mathcal{T}, \Phi)_i$ of each vertex $i \in V$ is exactly the cone angle at v_i.

4 Combinatorial Ricci flows in hyperbolic background geometry

In this section, we establish a compactness theorem for combinatorial Ricci flows on surfaces in hyperbolic background geometry, thus we can prove the main results in next section.

Given a weighted triangulated closed surface $(X, \mathcal{T}, \Phi)$ with $\Phi : E \to [0, \pi)$, if all the triangulations are isometric gluing by hyperbolic triangle, then we say that the triangulations have hyperbolic background geometry.

In this section, we always let $s(r_i(t)) = \sinh(r_i(t))$. Recall that for each triangle $\triangle ijk \in F$, θ_i, θ_j and θ_k are the inner angles at v_i, v_j and v_k, respectively.

Chow-Luo [6] proved some symmetric properties of circle packings.

Lemma 4.1 (Chow-Luo, [6]). *In hyperbolic background geometry $\mathbb{H}^2$, for all angles Φ_{ij}, Φ_{jk}, $\Phi_{ki} \in [0,\pi)$, if the triangle of radii r_i, r_j and r_k intersecting at these angles exists, then we have*

$$\frac{\partial \theta_i}{\partial r_j} s(r_j) = \frac{\partial \theta_j}{\partial r_i} s(r_i).$$

Proposition 4.1. *Suppose all the weight function Φ are in $[0,\pi)$. If the radii r_i evolves in time t according to the differential equation*

$$\frac{dr_i(t)}{dt} = -K_i s(r_i(t)),$$

where K_i is the discrete Gauss curvature at the vertex v_i, then the evolution of the inner angle θ_i satisfies the following equation

$$d\theta_i/dt = -A_{ij}(K_j - K_i) - A_{ik}(K_k - K_i) + A_i K_i,$$

where A_i and A_{rs} are positive valued elementary functions in r_i, r_j and r_k, for $r \neq s$. Furthermore, $A_{rs} = A_{sr}$.

Proof. By the chain rule, we have

$$\begin{aligned} \frac{d\theta_i}{dt} =& \frac{\partial \theta_i}{\partial r_i} r_i' + \frac{\partial \theta_i}{\partial r_j} r_j' + \frac{\partial \theta_i}{\partial r_k} r_k' \\ =& -\frac{\partial \theta_i}{\partial r_i} K_i s(r_i(t)) - \frac{\partial \theta_i}{\partial r_j} K_j s(r_j(t)) - \frac{\partial \theta_i}{\partial r_k} K_k s(r_k(t)) \\ =& -\frac{\partial \theta_i}{\partial r_j} s(r_j)(K_j - K_i) - \frac{\partial \theta_i}{\partial r_k} s(r_k)(K_k - K_i) - D_i K_i. \end{aligned}$$

Here,

$$D_i = \partial\theta_i/\partial r_i s(r_i) + \partial\theta_i/\partial r_j s(r_j) + \partial\theta_i/\partial r_k s(r_k).$$

By Lemma 4.1, we can write D_i as

$$\sum_{l\in\{i,j,k\}} \frac{\partial \theta_l}{\partial r_i} s(r_i) = s(r_i) \frac{\partial(\sum_{l\in\{i,j,k\}} \theta_l)}{\partial r_i}.$$

By Lemma 3.2, the last partial derivative is negative in the case of $\mathbb{H}^2$, define $A_i = -D_i > 0$, let $A_{mn} = \partial\theta_m/\partial r_n s(r_n)$ for $m \neq n$, then by Lemma 3.2 and Lemma 4.1, A_{mn} is a positive elementary function and $A_{mn} = A_{nm}$. Thus the result follows. $\square$

On triangulated closed surfaces, people have the following discrete version of Gauss-Bonnet formula [6], and this formula is useful in our later calculation.

Proposition 4.2 (Gauss-Bonnet). *Let $\lambda = -1, 0, 1$ be the curvature of one of the three geometries $\mathbb{H}^2$, $\mathbb{E}^2$, $\mathbb{S}^2$ used in the construction. If the circle packings metric based on a weighted generalized triangulation $(\mathcal{T}, \Phi)$ exists where $\Phi : E \to [0, \pi)$, then the total curvature*

$$\sum_{i=1}^{N} K_i = 2\pi\chi(X) - \lambda Area(X).$$

Proposition 4.3. *In hyperbolic background geometry, given a weighted triangulated closed surface $(X, \mathcal{T}, \Phi)$ with $\Phi : E \to [0, \pi)$, under the Ricci flow* (3.1)*, the curvature $K_i(t)$ evolves according to*

$$dK_i/dt = \sum_{j\sim i} C_{ij}(K_j - K_i) - B_i K_i,$$

where the sum is over all vertices v_j adjacent to v_i, C_{ij} and B_i are positive elementary functions in the radii $r_1, \cdots, r_n$. Furthermore, $C_{ij} = C_{ji}$.

Proof. Actually, this is a direct consequence of Proposition 4.1, since

$$dK_i/dt = -\sum_{j,k} d\theta_i^{jk}/dt,$$

where θ_i^{jk} are inner angles at v_i in the triangle $\triangle v_i v_j v_k$. □

As a consequence, similar to Chow-Luo's procedure in [6], we can obtain the discrete maximum principle for curvature evolution equation for obtuse exterior intersection angles $\Phi \in [0, \pi)$.

Theorem 4.1 (The Maximum Principle). *Given a weighted triangulated closed surface $(X, \mathcal{T}, \Phi)$ with $\Phi : E \to [0, \pi)$. In hyperbolic background geometry $\mathbb{H}^2$, suppose $r(t) = (r_1(t), \cdots, r_n(t))$ is the solution to combinatorial Ricci flow* (2.1) *in an interval. Let $M(t) = \max\{K_1(t), \cdots, K_n(t)\}$ and $m(t) = \min\{K_1(t), \cdots, K_n(t)\}$, then $\max(M(t), 0)$ is non-increasing in time t and $\min(m(t), 0)$ is non-decreasing in t.*

Applying the maximum principle, we obtain the following long time existence results.

Proposition 4.4. *Given a weighted triangulated closed surface $(X, \mathcal{T}, \Phi)$ with $\Phi : E \to [0, \pi)$. In hyperbolic background geometry $\mathbb{H}^2$, for any initial metric $r(0) \in \mathbb{R}^N_{>0}$, the solution to the Ricci flow* (2.1) *with the given initial metric exists for all time $t \geq 0$.*

Proof. We prove that as long as time t is bounded, then $r_i(t)$ and $1/r_i(t)$ remain bounded, so that the solution to combinatorial Ricci flow (2.1) exists for all time $t \geq 0$. Since

$$\frac{dr_i}{dt} = -K_i \sinh r_i,$$

we have

$$K_i(t) = d(\ln(\coth(r_i/2))),$$

then we have $\coth \frac{r_i}{2} \leq ce^{2\pi t}$, where c is some constant. Hence, $r_i(t)$ is bounded away from 0 as long as t is bounded. According to the Maximum principle Theorem 4.1, we have $r_i(t)$ is bounded from above as well, to see this, we first show the following lemma.

Lemma 4.2 (Chow-Luo, [6]). *For any $\epsilon > 0$, there exists a constant $C_2 = C_2(\epsilon)$ such that when $r_i > C_2$, the inner angle θ_i in hyperbolic triangle $\triangle v_i v_j v_k$ is smaller than ϵ.*

According to this lemma, if there exists a sequence $t_n \to \eta$, where $\eta \in (0, +\infty]$ such that $r_i(t_n) \to \infty$, then $K_i(t_n)$ will tend to 2π, but due to the discrete maximum principle, we have $M(t) < 2\pi$, which is a contradiction, so we complete the proof. $\square$

In our paper, one of the main tools is the long time convergence result of the combinatorial Ricci flows in hyperbolic background geometry.

Lemma 4.3. *Given a weighted triangulated closed surface $(X, \mathcal{T}, \Phi)$ with $\Phi : E \to [0, \pi)$. In hyperbolic background geometry $\mathbb{H}^2$, suppose $r(t)$ for $t \in [0, +\infty)$ is solution to Ricci flow* (2.1) *so that the set $\{r(t) | t \in (0, +\infty)\}$ lies in a compact region in $\mathbb{R}^N_{>0}$, then $r(t)$ converges exponentially fast to a circle packings metric in $\mathbb{R}^N_{>0}$ whose curvature at each vertex is zero.*

Proof. The proof is similar to Chow-Luo's proof of Proposition 3.7 in [6], the major change lies in the weight function, since we generalize the weight function from $\Phi : E \to [0, \pi/2]$ to $\Phi : E \to [0, \pi)$, but due to Lemma 3.2, we also derived the evolution equation and the maximum principle (see Proposition 4.3 and Theorem 4.1), so we also have this compactness theorem. $\square$

In order to get the lower bound estimate for combinatorial Ricci flows (2.1), we first show some comparison principles.

Lemma 4.4 (G. Lin, [12]). *Let $f : [0, +\infty) \to \mathbb{R}$ be a locally Lipschitz function. Suppose that there exists a constant C_1, such that $f'(t) \leq 0$ for a.e. $t \in \{f > C_1\}$, then*

$$f(t) \leq \max\{f(0), C_1\}, \quad \forall t \in [0, +\infty).$$

Similarly, if $f'(t) \geq 0$ for a.e. $t \in \{f < C_1\}$, then

$$f(t) \geq \min\{f(0), C_1\}, \quad \forall t \in [0, +\infty).$$

Lemma 4.5 (G. Lin, [12]). *Let $r(t)$ be a solution to combinatorial Ricci flow* (2.1). *Then there exists a positive constant $C_3 = C_3(\mathcal{T}, r(0)) > 0$ depending on the triangulation $\mathcal{T}$ and initial data $r(0)$, such that $r_i(t) \leq C_3$ for all $i \in V$.*

For a hyperbolic triangle $\triangle ijk$ in $\mathbb{H}^2$, let r_i, r_j, r_k be the radius of the circles centered at vertex v_i, v_j, v_k. We denote the interior angle at each vertex v_i by $\theta_i = \theta_i(\vec{r})$, where $\vec{r} = (r_i, r_j, r_k)$.

Lemma 4.6. *Given a weighted triangulated closed surface $(X, \mathcal{T}, \Phi)$ with $\Phi : E \to [0, \pi)$. If for each three-circle packings configuration r_i, r_j, r_k and $\Phi_{ij}, \Phi_{jk}, \Phi_{ki}$, assume that $\lambda_{ijk} \geq 0$, $\lambda_{jki} \geq 0$ and $\lambda_{kij} \geq 0$, then the angle function $t \mapsto \theta_i(t\vec{1})$ is continuously differentiable and strictly decreasing in $(0, +\infty)$.*

Proof. When $\Phi \in [0, \pi)$, it is obvious that $\cos\Phi_{ij}$, $\cos\Phi_{jk}$ and $\cos\Phi_{ki} \in (-1, 1]$. Similar to Ge-Lin's proof in [12], set $f(t) = \cos\theta_i(t\vec{1})$, and by direct computation,

$$f(t) = \frac{(1+\cos\Phi_{ij})(1+\cos\Phi_{ki})\sinh^2 t + (1+\cos\Phi_{ij}+\cos\Phi_{ki}-\cos\Phi_{jk})}{f_1(t)f_2(t)},$$

where

$$\begin{aligned} f_1(t) &= \sqrt{(1+\cos\Phi_{ij})^2\sinh^2 t + 2(1+\cos\Phi_{ij})}, \\ f_2(t) &= \sqrt{(1+\cos\Phi_{ki})^2\sinh^2 t + 2(1+\cos\Phi_{ki})}. \end{aligned}$$

Then the derivative of $f(t)$ is

$$f'(t) = \frac{a(\Phi)\sinh(2t)[b(\Phi)\sinh^2 t + c(\Phi)]}{f_1^3(t)f_2^3(t)},$$

where

$$\begin{aligned} a(\Phi) &= (1+\cos\Phi_{ij})(1+\cos\Phi_{ki}), \\ b(\Phi) &= (1+\cos\Phi_{ij})(1+\cos\Phi_{ki})(1+\cos\Phi_{jk}), \\ c(\Phi) &= (1+\cos\Phi_{jk})(2+\cos\Phi_{ij}+\cos\Phi_{ik}) - (\cos\Phi_{ij}-\cos\Phi_{ki})^2, \end{aligned}$$

by the same calculations, we also have $f_1(t) > 0$, $f_2(t) > 0$, $a(\Phi) > 0$, $b(\Phi) > 0$, the difference lies in $c(\Phi)$, in our case, since

$$\begin{cases} \lambda_{ijk} = \cos\Phi_{ij} + \cos\Phi_{jk}\cos\Phi_{ki} \geq 0, \\ \lambda_{jki} = \cos\Phi_{jk} + \cos\Phi_{ki}\cos\Phi_{ij} \geq 0, \\ \lambda_{kij} = \cos\Phi_{ki} + \cos\Phi_{ij}\cos\Phi_{jk} \geq 0, \end{cases}$$

so we have

$$
\begin{aligned}
c(\Phi) =&(1+\cos\Phi_{jk})(2+\cos\Phi_{ij}+\cos\Phi_{ki})-(\cos\Phi_{ij}-\cos\Phi_{ki})^2\\
=&(2-\cos^2\Phi_{ij}-\cos^2\Phi_{ki})+(\cos\Phi_{ij}+\cos\Phi_{jk}\cos\Phi_{ki})\\
&+(\cos\Phi_{ki}+\cos\Phi_{ij}\cos\Phi_{jk})+2(\cos\Phi_{jk}+\cos\Phi_{ki}\cos\Phi_{ij})>0.
\end{aligned}
$$

Therefore, $f'(t)>0$, that is, $f(t)$ is strictly increasing, which is equivalent to that $\theta_i(t\vec{1})$ is strictly decreasing. □

Lemma 4.7. *For any weight function $\Phi: E\to[0,\pi)$, the following limits exist:*

$$
\lim_{t\to 0}\theta_i(t\vec{1})=\arccos\left(\frac{1+\cos\Phi_{ij}+\cos\Phi_{ki}-\cos\Phi_{jk}}{2\sqrt{1+\cos\Phi_{ij}}\sqrt{1+\cos\Phi_{ki}}}\right),
$$

where $\vec{1}=(1,1,1)$ and $\lim_{t\to+\infty}\theta_i(t\vec{1})=1$. As a special case, if the weight $\Phi: E\to[0,\pi)$ is a constant, we obtain that $\lim_{t\to 0}\theta_i(t\vec{1})=\frac{\pi}{3}$.

Proof. Taking the limits directly, we have

$$
\begin{aligned}
\lim_{t\to 0}\cos\theta_i(t\vec{1}) =&\lim_{t\to 0}\frac{(1+\cos\Phi_{ij})(1+\cos\Phi_{ki})\sinh^2 t+(1+\cos\Phi_{ij}+\cos\Phi_{ki}-\cos\Phi_{jk})}{f_1(t)f_2(t)}\\
=&\frac{1+\cos\Phi_{ij}+\cos\Phi_{ki}-\cos\Phi_{jk}}{2\sqrt{1+\cos\Phi_{ij}}\sqrt{1+\cos\Phi_{ki}}}.
\end{aligned}
$$

So

$$
\lim_{t\to 0}\theta_i(t\vec{1})=\arccos\left(\frac{1+\cos\Phi_{ij}+\cos\Phi_{ki}-\cos\Phi_{jk}}{2\sqrt{1+\cos\Phi_{ij}}\sqrt{1+\cos\Phi_{ki}}}\right).
$$

If the weight $\Phi: E\to[0,\pi)$ is a constant, that is, $\Phi_{ij}=\Phi_{jk}=\Phi_{ki}$, then we have

$$
\lim_{t\to 0}\theta_i(t\vec{1})=\arccos\frac{1}{2}=\frac{\pi}{3}.
$$

This completes the proof. □

Proposition 4.5. *Given any weight function $\Phi: E\to[0,\pi)$, for a fixed hyperbolic triangle $\triangle v_iv_jv_k$, if $r_i=\min\{r_i,r_j,r_k\}$, then $\theta_i(\vec{r})\geq\theta_i(r_i\vec{1})$, if $r_i=\max\{r_i,r_j,r_k\}$, then $\theta_i(\vec{r})\leq\theta_i(r_i\vec{1})$.*

Proof. Given a weight function $\Phi: E\to[0,\pi)$, by Lemma 3.2, we have

$$
\frac{\partial\theta_i}{\partial r_j}\geq 0,\quad \forall i\neq j.
$$

This is the key point, then by the same argument in Ge-Lin [12, Lemma 5.6], let $\sigma : [0,1] \to \mathbb{R}^3_{>0}$ defined as

$$\sigma(s) = (r_i(s), r_j(s), r_k(s)) := (1-s)\vec{r} + sr_i\vec{1},$$

consider

$$r_i(s) = r_i, \quad r_j(s) = r_j + s(r_i - r_j), \quad r_k(s) = r_k + s(r_i - r_k), \quad \forall s \in [0,1].$$

If $r_i = \min\{r_i, r_j, r_k\}$, then by Lemma 3.2,

$$\frac{d}{ds}(\theta_i(\sigma(s)) = (r_i - r_j)\frac{\partial \theta_i}{\partial r_j}(\sigma(s)) + (r_i - r_k)\frac{\partial \theta_i}{\partial r_k}(\sigma(s)) < 0,$$

so $\theta_i(\sigma(s))$ is decreasing in $[0,1]$, hence,

$$\theta_i(\vec{r}) \geq \theta_i(r_i\vec{1}).$$

As for the case $r_i = \max\{r_i, r_j, r_k\}$, the argument is similar. □

5 Proof of the main results

In this section, we prove the main results. First, we establish the following lower bound estimate for the solution to the combinatorial Ricci flow (2.1).

Theorem 5.1. *Given a weighted triangulated closed surface $(X, \mathcal{T}, \Phi)$ with $\Phi : E \to [0,\pi)$, assume that the character $\mathcal{L}(\mathcal{T}, \Phi)_i > 2\pi$ for all $i \in V$. Let $r(t)$ be a solution to the combinatorial Ricci flow* (2.1)*, then there exists a positive constant $C = C(\mathcal{T}, \Phi, r(0)) > 0$ depending only on the weighted triangulation $(\mathcal{T}, \Phi)$ and the initial data $r(0)$, such that $r_i(t) \geq C$ for all $i \in V$ and all $t > 0$.*

Proof. First, we construct a locally Lipschitz function $g(t) := \min_{m\in V} r_m(t)$, then for a.e. $t \in [0, +\infty)$, there exists a special vertex $i \in V$ depending on t, such that $g(t) = r_i(t)$, $g'(t) = {r_i}'(t)$. By Lemma 4.6 and Lemma 4.7, $\theta_i(t\vec{1})$ is continuously differentiable for all $i \in V$, and

$$\lim_{t\to 0}\theta_i(t\vec{1}) = \arccos\left(\frac{1 + \cos\Phi_{ij} + \cos\Phi_{ki} - \cos\Phi_{jk}}{2\sqrt{1+\cos\Phi_{ij}}\sqrt{1+\cos\Phi_{ki}}}\right).$$

That is, $\forall \epsilon > 0$, there exists a constant $C = C(\mathcal{T}, \Phi, r(0)) > 0$, such that $\forall t \leq C$,

$$\theta_i(t\vec{1}) \geq \arccos\left(\frac{1 + \cos\Phi_{ij} + \cos\Phi_{ki} - \cos\Phi_{jk}}{2\sqrt{1+\cos\Phi_{ij}}\sqrt{1+\cos\Phi_{ki}}}\right) - \epsilon.$$

Now, according to the same argument of Ge-Lin [12], we have $g'(t) \geq 0$ for a.e. $t \in \{g < C\}$. Actually, let $t \in \{g < C\}$ and $t \in [0, +\infty)$, for any hyperbolic triangle $\triangle ijk$ incident to i realized by the circle packings $r(t)$, by the definition of $g(t) = \min_{m \in V} r_m(t)$, so $r_i(t) = \min\{r_i(t), r_j(t), r_k(t)\} < C$. Let $\vec{r}(t) = (r_i(t), r_j(t), r_k(t))$, then by Lemma 4.5,

$$\theta_i(\vec{r}(t)) \geq \theta_i(r_i(t)\vec{1}) \geq \arccos\Big(\frac{1 + \cos\Phi_{ij} + \cos\Phi_{ki} - \cos\Phi_{jk}}{2\sqrt{1 + \cos\Phi_{ij}}\sqrt{1 + \cos\Phi_{ki}}}\Big) - \epsilon.$$

Due to the assumption that the character $\mathcal{L}(\mathcal{T}, \Phi)_j > 2\pi$ for all $j \in V$, let

$$0 < \epsilon < \frac{\min_{j \in V}(\mathcal{L}(\mathcal{T}, \Phi)_j - 2\pi)}{\max_{j \in V} d_j},$$

then we have

$$\begin{aligned}
K_i(r(t)) =& 2\pi - \sum_{\triangle ijk \in F} \theta_i(\vec{r}(t)) \\
\leq& 2\pi - \sum_{\triangle ijk \in F} \Big(\arccos\Big(\frac{1 + \cos\Phi_{ij} + \cos\Phi_{ki} - \cos\Phi_{jk}}{2\sqrt{1 + \cos\Phi_{ij}}\sqrt{1 + \cos\Phi_{ki}}}\Big) - \epsilon\Big) \\
=& \epsilon \cdot d_i - (\mathcal{L}(\mathcal{T}, \Phi)_i - 2\pi) \\
\leq& \epsilon \cdot \max_{j \in V} d_j - \min_{j \in V}(\mathcal{L}(\mathcal{T}, \Phi)_j - 2\pi) < 0.
\end{aligned}$$

Since $g(t) = r_i(t)$, $g'(t) = {r_i}'(t)$, so we have

$$g'(t) = {r_i}'(t) = -K_i \sinh r_i > 0.$$

Hence, $g'(t) > 0$ for a.e. $t \in \{g < C\}$. By Proposition 4.5, we have $g(t) \geq \min\{g(0), C\}$, $\forall t \in [0, +\infty)$. So there exists a positive constant $C = C(\mathcal{T}, \Phi, r(0)) > 0$, such that $r_i(t) \geq C$ for all vertices and all times, which complete the proof. $\square$

Let $r(t)$ be the solution to combinatorial Ricci flows, then by Theorem 5.1 and Proposition 4.3, there exists a positive constant C_1 and C_2 depending on the weighted triangulation $(\mathcal{T}, \Phi)$ and the initial data $r(0)$, such that

$$C_1 \leq r_i(t) \leq C_2, \quad \forall i \in V, \quad t \in [0, +\infty).$$

This is equivalent to say that $r(t)$ lies in the compact region in $\mathbb{R}^N_{>0}$. Then **the existence part** of Theorem 1.3 is obtained by Proposition 1.1.

On the other hand, **the uniqueness part** of Theorem 1.3 is a consequence of the rigidity part of Koebe-Andreev-Thurston theorem, i.e., the curvature map $r \mapsto K$ is injective.

Now, we prove **the nonexistence part** of Theorem 1.3.

Proof. Assume $\mathcal{L}(\mathcal{T},\Phi)_i \leq 2\pi$ at each vertex $i \in V$. Hence the average character $\mathcal{L}_{av} \leq 2\pi$. By Proposition 3.4 in [12], we get $\chi(X) \geq 0$. We denote the hyperbolic area of a hyperbolic triangle $\triangle v_i v_j v_k$ by $Area(\triangle v_i v_j v_k)$, so

$$Area(X) = \sum_{\triangle v_i v_j v_k \in F} Area(\triangle v_i v_j v_k).$$

By the discrete Gauss-Bonnet formula in hyperbolic background geometry,

$$\sum_{i\in V} K_i = 2\pi\chi(X) + Area(X).$$

Hence, there is no circle packings metric r with curvature zero. Since Chow-Luo's result [6] show that combinatorial Ricci flows converges if and only if there exists a circle packings metric r_{ze} with curvature zero, therefore, we get the nonexistence part of Theorem 1.3. □

On the other hand, if $\mathcal{L}(\mathcal{T},\Phi)_i < 2\pi$ for all vertices, we can even show that any initial circle packings shrinks to a point along combinatorial Ricci flow.

Theorem 5.2. *If the character $\mathcal{L}(T,\Phi)_i < 2\pi$ or $\mathcal{L}(T,\Phi)_i \leq 2\pi$ for a constant weight at each vertex, then the solution $r(t)$ to the combinatorial Ricci flow* (2.1) *satisfies $r(t) \to 0$ when t tends to $+\infty$.*

Now, we use Theorem 1.3 to prove Theorem 1.4.

Proof. If the character $\mathcal{L}(\mathcal{T},\Phi)_i > 2\pi$ at all vertices, by part (a) of Theorem 1.3, the solution $r(t)$ to the combinatorial Ricci flow (2.1) converges to a unique hyperbolic circle packings r_{ze} which determines a unique complete hyperbolic metric μ on X.

If the character $\mathcal{L}(\mathcal{T},\Phi)_i \leq 2\pi$ at all vertices, by part (b) of Theorem 1.3, any solution $r(t)$ to the flow (2.1) satisfies $r(t) \to 0$ which means that $r(t)$ can not converges when t tends to $+\infty$.

Finally, since we have generalized the compactness theorem for the combinatorial Ricci flow (2.1) in hyperbolic background geometry with weight function $\Phi \in [0,\pi)$, then the solution $r(t)$ to the combinatorial Ricci flow (2.1) converges if and only if there exists a hyperbolic circle packings. Therefore, the proof is completed. □

Then, we can use Theorem 1.4 to prove Theorem 1.1 and Theorem 1.2.

Proof of Theorem 1.1. For any given constant weight $\Phi : E \to [0,\pi)$, if $d_i \geq 7$ at any vertex $i \in V$, then by the definition of $\mathcal{L}(\mathcal{T},\Phi)_i$, we have

$$\mathcal{L}(\mathcal{T},\Phi)_i = \sum_{\triangle ijk \in F} \arccos\frac{1}{2} = \frac{\pi}{3}d_i \geq \frac{7\pi}{3},$$

so $\mathcal{L}(\mathcal{T},\Phi)_i > 2\pi$ at all vertices, by Theorem 1.3, there exists a unique complete hyperbolic metric μ on X so that (X,μ) supports a $(\mathcal{T},\Phi)$-type circle packings $\mathcal{P}$. □

Proof of Theorem 1.2. For any weight $\Phi \in [\arccos\eta, \arccos\xi] \subset [\pi/2, \pi)$, where $-1 < \xi \leq \eta \leq 0$ are arbitrary chosen so that $\eta < (2\cos\frac{2\pi}{7} - 1 + \xi)/(2 - 2\cos\frac{2\pi}{7})$, let $k = 2\cos\frac{2\pi}{7} > 0$, $\forall a, b \in (-1, 0]$, by Ge-Lin [12, Proposition 3.6],

$$1 + a + b - k\sqrt{1+a}\sqrt{1+b} \leq 1 + 2\eta - k(1+\eta),$$

since $\eta < (k - 1 + \xi)/(2-k)$, that is $1 + 2\eta - k(1+\eta) < \xi$, then we have

$$1 + a + b - k\sqrt{1+a}\sqrt{1+b} < \xi.$$

For any given weight $\Phi \in [\arccos\eta, \arccos\xi] \subset [\pi/2, \pi)$, we have $\cos\Phi_{ij}, \cos\Phi_{jk}, \cos\Phi_{ki} \in [\xi, \eta]$, by the above inequality,

$$1 + \cos\Phi_{ij} + \cos\Phi_{ki} - k\sqrt{1+\cos\Phi_{ij}}\sqrt{1+\cos\Phi_{ki}} < \xi \leq \cos\Phi_{jk}.$$

Then

$$\frac{1 + \cos\Phi_{ij} + \cos\Phi_{ki} - \cos\Phi_{jk}}{2\sqrt{1+\cos\Phi_{ij}}\sqrt{1+\cos\Phi_{ki}}} < \frac{k}{2} = \cos\frac{2\pi}{7},$$

that is,

$$\arccos\frac{1 + \cos\Phi_{ij} + \cos\Phi_{ki} - \cos\Phi_{jk}}{2\sqrt{1+\cos\Phi_{ij}}\sqrt{1+\cos\Phi_{ki}}} > \frac{2\pi}{7}.$$

As a consequence, if $d_i \geq 7$, then

$$\mathcal{L}(\mathcal{T}, \Phi)_i = \sum_{\triangle ijk \in F} \frac{2\pi}{7} = 7 \cdot \frac{2\pi}{7} \geq 2\pi.$$

By Theorem 1.3, there exists a unique complete hyperbolic metric μ on X so that (X, μ) supports a $(\mathcal{T}, \Phi)$-type circle packings $\mathcal{P}$. When $\Phi \in [\arccos\eta, \arccos\xi] \subset [0, \pi/2]$, the proof is similar. $\square$

Remark 5.1. In the acute angle case, Ge-Lin [12, Theorem 2.3], proved that for a closed surface X, if X admits a triangulation $\mathcal{T}$ with degree $d > 9$ at each vertex, then for any given weight $\Phi : E \to [0, \pi/2]$, there exists a unique complete hyperbolic metric μ on X such that (X, μ) supports a (T, Φ)-type circle packings P. But when $\Phi \in [0, \pi)$, we do not have similar result. In fact, we can choose some $\Phi_{ij}, \Phi_{jk}, \Phi_{ki}$ which satisfies $\lambda_{ijk}, \lambda_{jki}, \lambda_{kij} \geq 0$, and

$$\frac{1 + \cos\Phi_{ij} + \cos\Phi_{ki} - \cos\Phi_{jk}}{2\sqrt{1+\cos\Phi_{ij}}\sqrt{1+\cos\Phi_{ki}}} \to 1,$$

for example, let $(\Phi_{ij}, \Phi_{jk}, \Phi_{ki}) = (1, 0.99, -0.99)$, then

$$\frac{1 + \cos\Phi_{ij} + \cos\Phi_{ki} - \cos\Phi_{jk}}{2\sqrt{1+\cos\Phi_{ij}}\sqrt{1+\cos\Phi_{ki}}} \approx 0.9975,$$

as a consequence, we can not choose a proper degree d_i, such that for any given weight, $\mathcal{L}(\mathcal{T}, \Phi)_i > 2\pi$ at each vertex.

Acknowledgements

The author is grateful to Professor Gang Tian for constant encouragement, support and guidance.

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