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On Stationary $p(\cdot)$-Kirchhoff-Type Problem with Mixed Boundary Conditions

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Abstract. In this paper, we consider the stationary Kirchhoff-type equation with mixed boundary conditions in a variable exponent Sobolev space. We show the existence of a weak solution. Moreover, we prove the existence of a non-trivial weak solution and infinitely many weak solutions under some hypotheses by variational methods.

Key Words: Mixed boundary value problem, $p(\cdot)$-Kirchhoff-type problem, existence of a weak solution, variational methods.

AMS Subject Classifications: 35J65, 35J20, 47J10

1 Introduction

In recent years, the Kirchhoff-type problems has been considered by many authors. The original Kirchhoff equation is as follows

$$\rho \frac{\partial^2 u}{\partial s^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right| dx \right) \frac{\partial^2 u}{\partial x^2} = 0,$$

where s denotes the time, ρ is the mass density, ρ_0 is the initial tension, h is the area of the cross section, E is the Young modulus of the material and L is the length of the string. We refer to Kirchhoff [25] and Lions [27] for physics. Such type of system is an extension of the classical D'Alambert wave equation, by considering the effects of the changes in the length of the string during the vibration. For some interesting results, see Arocio and Pannizi [9], Cavalcante et al. [10], Corrêa and Figueiredo [13], D'Ancona and Spagnolo [14], and He and Zou [24].

The stationary analogue of the Kirchhoff equation with the Dirichlet boundary condition takes the form

$$\begin{cases} -\left(a + b \int_\Omega |\nabla u|^2 dx \right) \Delta u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma, \end{cases}$$

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where a and b are positive constants.

In this paper, we are interested in the following stationary Kirchhoff-type problem with mixed boundary conditions

$$\begin{cases} -M\left(\frac{1}{2}\int_\Omega S(x,|\nabla u|^2)dx\right)\mathrm{div}\,[S_t(x,|\nabla u|^2)\nabla u]=f(x,u) & \text{in } \Omega,\\ u=0 & \text{on } \Gamma_1,\\ -M\left(\frac{1}{2}\int_\Omega S(x,|\nabla u|^2)dx\right)S_t(x,|\nabla u|^2)\frac{\partial u}{\partial \boldsymbol{n}}=g(x,u) & \text{on } \Gamma_2. \end{cases} \tag{1.1}$$

Here Ω is a bounded domain in $\mathbb{R}^d$ $(d\ge 2)$ with a Lipschitz-continuous ($C^{0,1}$ shortly) boundary Γ, and Γ_1 and Γ_2 are disjoint open subsets of Γ such that $\overline{\Gamma}_1\cup\overline{\Gamma}_2=\Gamma$ and $\boldsymbol{n}$ denotes the unit outer normal vector to the boundary Γ. The Kirchhoff function $M:[0,\infty)\to[0,\infty)$ is continuous and a function $S(x,t)$ defined in $\Omega\times[0,\infty)$ depends on a variable exponent $p(x)$ and satisfies some structure conditions. The functions $f(x,t)$ and $g(x,t)$ are given functions defined on $\Omega\times\mathbb{R}$ and $\Gamma_2\times\mathbb{R}$, respectively.

In Yücedag [35], the author got the existence and multiplicity of weak solutions for extended system which contains an operator div $[a(x,|\nabla u|)$ in stead of div $[S_t(x,|\nabla u|^2)\nabla u]$ in (1.1) containing $p(\cdot)$-Laplace operator

$$\Delta_{p(x)}u=\mathrm{div}\,[|\nabla u|^{p(x)-2}\nabla u],$$

where $p(x)>1$ with the Dirichlet boundary condition. In particular case where $M\equiv 1$ with the Dirichlet or the Nemann boundary condition, there are many works, see Figueiredo et al. [15], Dong and Xu [18], Wang [31], Zhang and Wang [38], Yuan and Du [34], Colasnonno and Noris [12], Torné [30], and for mixed boundary condition, see Aramaki [1,3], Arena et al. [7], and Ge and Tian [23]. For mathematical physics of electrorheological fluid, see Diening [16] and Růžička [28].

In the present paper, we attempt the other extension to more general system than $p(\cdot)$-Laplace operator with the mixed boundary values. This causes some difficulties in calculations and requires more general conditions. To our best knowledge, the present paper are not covered in the literature.

The paper is organized as follows. Section 2 consists of three subsections. In subsection 2.1, we introduce variable exponent Lebesgue and Sobolev spaces and their properties. In subsection 2.2, we define a Carathéodory function $S(x,t)$ having some structure conditions and its properties. In subsection 2.3, we introduce the spaces of functions used in this paper. The propositions stated in this subsection are crucial for this paper and they seem to be new. Section 3 is devoted to state some main theorems and their proofs.

2 Preliminaries

Let Ω be a bounded domain in $\mathbb{R}^d$ $(d\ge 2)$ with a Lipschitz-continuous (shortly $C^{0,1}$) boundary Γ. Moreover, we assume that Γ_1 and Γ_2 are disjoint open subsets of Γ such that

$$\overline{\Gamma}_1\cup\overline{\Gamma}_2=\Gamma \quad\text{and}\quad \Gamma_1\neq\emptyset. \tag{2.1}$$