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An Inverse Operator Approach to a Fractional Nonlinear Integro-Differential Equation

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Abstract. This paper studies the existence, uniqueness and stability for a new fractional nonlinear integro-differential equation with an integral boundary condition using several well-known fixed point theorems in a Banach space. The method used is to convert the equation into an equivalent implicit integral equation based on a bounded inverse operator which is an infinite series and uniformly convergent. Furthermore, we compute approximate values of a few multivariate Mittag-Leffler functions by our Python codes in illustrative examples demonstrating the use of key theorems derived. These investigations have a wide range of applications as existence, uniqueness and stability often appear in various pure and applied research areas.

Key Words: Fractional nonlinear integro-differential equation, uniqueness and existence, fixed point theory, multivariate Mittag-Leffler function, inverse operator.

AMS Subject Classifications: 34B15, 34A12, 34K20, 26A33

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1 Introduction and preliminaries

We begin introducing some basic definitions from fractional analysis, the multivariate Mittag–Leffler function, a bounded inverse operator for solving a fractional nonlinear equation with a nonlocal initial condition, as well as the current work on fractional nonlinear integro-differential equations.

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Let $\mathcal{T} > 0$. The Riemann-Liouville fractional integral I^β of order $\beta \in \mathbb{R}^+$ is defined for the function Ψ [1,2] as

$$(I^\beta \Psi)(\zeta) = \frac{\zeta^{\beta-1}}{\Gamma(\beta)} * \Psi = \frac{1}{\Gamma(\beta)} \int_0^\zeta (\zeta-\tau)^{\beta-1}\Psi(\tau)d\tau, \quad \zeta \in [0,\mathcal{T}],$$

where $*$ is the convolution given by

$$(f_1 * f_2)(\zeta) = (f_2 * f_1)(\zeta) = \int_0^\zeta f(\zeta-\tau)g(\tau)d\tau = \int_0^\zeta f(\tau)g(\zeta-\tau)d\tau.$$

In particular from [3],

$$I^0\phi = \delta * \phi = \phi,$$

where δ is the delta distribution in the Schwartz sense.

The Liouville-Caputo fractional derivative ${}_CD^{\beta_1}$ of order $\beta_1 \in (0,1]$ of the function $\Psi(\zeta)$ is defined as [1]

$$({}_CD^{\beta_1}\Psi)(\zeta) = \Big(I^{1-\beta_1}\frac{d}{dx}\Psi\Big)(\zeta) = \frac{1}{\Gamma(1-\beta_1)} \int_0^\zeta (\zeta-\tau)^{-\beta_1}\Psi'(\tau)d\tau, \ \ \zeta \in [0,\mathcal{T}].$$

The Banach space $C[0,\mathcal{T}]$ consists of all continuous mappings from $[0,\mathcal{T}]$ into $\mathbb{R}$ with the norm

$$\|\Psi\| = \max_{\zeta\in[0,\mathcal{T}]} |\Psi(\zeta)| < +\infty.$$

It follows that for $\Psi \in C[0,\mathcal{T}]$

$$\|I^\beta\Psi\| = \max_{\zeta\in[0,\mathcal{T}]} \left| \frac{1}{\Gamma(\beta)} \int_0^\zeta (\zeta-\tau)^{\beta-1}\Psi(\tau)d\tau \right| \le \frac{\mathcal{T}^\beta}{\Gamma(\beta+1)}\|\Psi\|,$$

which implies

$$\|I^\beta\| \le \frac{\mathcal{T}^\beta}{\Gamma(\beta+1)}.$$

Stability analysis is essentially important to differential equations since it guarantees that slight derivations from mathematical model caused by errors have a correspondingly slight effect on the solution, so that the equation describing the model will accurately predict the future outcomes. In addition, when considering a system given by differential equations, most of the time we are unable to find exact solutions. However, approximate solutions could serve our purpose in general if the system is stable.