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A Local Model Reduction Method Based on *k*-Nearest-Neighbors for Parametrized Nonlocal Problems

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Abstract. In this paper, the model reduction method based on *k*-nearest-neighbors is provided for the parametrized nonlocal partial differential equations (PDEs). In comparison to standard local PDEs, the stiffness matrix of the corresponding nonlocal model loses sparsity due to the nonlocal interaction parameter δ. Specially the nonlocal model contains uncertain parameters, enhancing the complexity of computation. In order to improve the computation efficiency, we combine the *k*-nearest-neighbors with the model reduction method to construct the efficient surrogate models of the parametrized nonlocal problems. This method is an offline-online mechanism. In the offline phase, we develop the full-order model by using the quadratic finite element method (FEM) to generate snapshots and employ the model reduction method to process the snapshots and extract their key characters. In the online phase, we utilize *k*-nearest-neighbors regression to construct the surrogate model. In the numerical experiments, we first verify the convergence rate when applying quadratic FEM to the nonlocal problems. Subsequently, for the linear and nonlinear nonlocal problems with random inputs, the numerical results illustrate the efficiency and accuracy of the surrogate models.

AMS subject classifications: 45A05, 45G10, 45P05, 65C30, 65R20, 65R99

Key words: Parametrized nonlocal PDEs, surrogate model, quadratic finite element method, proper orthogonal decomposition, dynamic mode decomposition, *k*-nearest-neighbors.

1 Introduction

Nonlocal phenomena, which have appeared observed in various fields such as physics, materials, biology, and social sciences, are ubiquitous in nature [11]. It is well-known

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that the nonlocal parabolic model is a generalization of the classic local parabolic model, incorporating the horizon parameter δ to measure the range of nonlocal interactions. The specific nonlocal parabolic equation is given by:

$$\frac{\partial u}{\partial t}=\mathcal{L}_\delta u+f.$$

As $\delta\to 0$, the nonlocal model converges to the corresponding classic local model. Essentially, the discretization of the local Laplacian operator only requires computation of the discrete node values, while the discretization of the nonlocal diffusion operator necessitates information on feature interactions occurring between spatial points separated by a finite distance when $\delta>0$ [14]. Consequently, the nonlocal model has significant advantages in exploring defects, for example, the nucleation and propagation of cracks [2].

In recent years, the exploration of theoretical and numerical analysis of nonlocal models has received much attention. Tian and Du [38] conducted a comprehensive study of piecewise constant finite element method (FEM) and piecewise linear FEM schemes for nonlocal diffusion and linear peridynamic equations, providing detailed discussions of singular kernels and analyzing fundamental theoretical and numerical scheme issues. Additionally, Li et al. [12, 15] investigated the exponential time differencing method for semilinear parabolic equations, such as the nonlocal Allen-Cahn equation, to explore its maximum bound principle. They also considered the nonlocal Cahn-Hilliard equation, rigorously establishing energy stability and convergence analysis [10,25]. In [13], the authors adopted the implicit Runge-Kutta method and discontinuous Galerkin method for nonlocal diffusion problems, where the stability and error estimates of the fully discrete numerical schemes were presented. Furthermore, Nan and Song [31] applied integrating factor Runge-Kutta method and finite difference method (FDM) to solve the nonlocal Allen-Cahn equation, successfully demonstrating the maximum bound principle.

It is well known that nonlocal interactions lead to a denser discrete system compared to standard local models due to the horizon δ, whether applying the FEM or the FDM. This increased density poses a significant challenge for the computation of the nonlocal model, and the inclusion of stochastic input further effects computational efficiency [24, 40]. Consequently, constructing a cheaper and simplified surrogate model for parametrized nonlocal model has become a significant subject of research. Surrogate model is an approximate model in the low dimensional subspace of the solution space. The success of these model reduction methods relies on the assumption that the solution manifold can be embedded in a low dimensional space [3]. But, the important class of problems arising from parametric dynamical systems typically induce a rough solution manifold with slowly decaying Kolmogorov n-widths. This suggests that traditional model-order reduction (MOR) methods [5, 22, 26, 28] are generally ineffective. In recent years, there has been a growing interest in the development of MOR techniques for parametric dynamical systems to overcome the limitations of linear global approximations. A large class of methods consider the dynamical low approximation, enabling both the deterministic and stochastic basis functions to evolve in time [30,32,45]. Other strategies,