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A Modified Convolution Quadrature Combined with the Method of Fundamental Solutions and Galerkin BEM for Acoustic Scattering

Ebraheem Aldahham[1,*] and Lehel Banjai[1]

[1] *Maxwell Institute for Mathematical Sciences, School of Mathematical and Computer Sciences, Heriot-Watt University, Edinburgh EH14 4AS, UK.*

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Abstract. We describe a numerical method for the solution of acoustic exterior scattering problems based on the time-domain boundary integral representation of the solution. As the spatial discretization of the resulting time-domain boundary integral equation we use either the method of fundamental solutions (MFS) or the Galerkin boundary element method (BEM). In time we apply either a standard convolution quadrature (CQ) based on an A-stable linear multistep method or a modified CQ scheme. It is well-known that the standard low-order CQ schemes for hyperbolic problems suffer from strong dissipation and dispersion properties. The modified scheme is designed to avoid these properties. We give a careful description of the modified scheme and its implementation with differences due to different spatial discretizations highlighted. Numerous numerical experiments illustrate the effectiveness of the modified scheme and dramatic improvement with errors up to two orders of magnitude smaller in comparison with the standard scheme.

AMS subject classifications: 45E10, 65M80, 65L60, 65T50

Key words: Acoustic wave scattering, convolution quadrature, modified convolution quadrature, method of fundamental solutions, boundary integral equation.

1 Introduction

In this paper we investigate a class of numerical methods for the scattering problem: Find $u(t) \in H^1(\Omega^+)$ such that

$$\begin{aligned} \partial_t^2 u - \Delta u &= 0 \quad \text{for } (t,x) \in [0,T] \times \Omega^+, \\ u(0) = \partial_t u(0) &= 0 \quad \text{for } x \in \Omega^+, \\ u &= g \quad \text{for } (t,x) \in [0,T] \times \Gamma, \end{aligned} \tag{1.1}$$

*Corresponding author. *Email addresses:* ea56@hw.ac.uk (E. Aldahham), l.banjai@hw.ac.uk (L. Banjai)

where $\Omega\subset\mathbb{R}^d$, $d=2,3$, is an open, bounded Lipschitz domain with exterior $\Omega^+=\mathbb{R}^d\setminus\overline{\Omega}$ and boundary $\Gamma=\partial\Omega$. If g is the trace of an incident wave $-u^{\text{inc}}$ on Γ with $\operatorname{supp} u^{\text{inc}}(0)\subset\Omega^+$ then u is the *scattered wave* and $u^{\text{tot}}=u+u^{\text{inc}}$ is the *total wave* scattered by the obstacle under the sound-soft boundary condition $u^{\text{tot}}|_\Gamma=0$.

There are a number of ways to tackle problem (1.1). A popular option is to consider the problem on a bounded domain containing Γ facilitated by an introduction of a transparent boundary condition. The exact transparent boundary condition is non-local in time and space and can be computed fast on special domains such as balls [1,24,26,30]. Alternatively, an approximate local boundary condition can be used, such as local absorbing boundary condition [20,27], methods based on the pole condition [32,33] and perfectly matched layers [14]. All of these methods apply to convex domains of a special shape, e.g., circle or rectangle in 2D. This can result in an unnecessarily expensive method if Γ is such that a very large convex domain is needed to encompass it, e.g., the elongated shape of an airplane. In such situations methods based on time-domain boundary integral equations are of advantage [10,17,35]. This particularly holds when the accurate computation of the far field potential is needed as the boundary integral potentials have the exact far-field behaviour encoded in their kernels.

Thus in this work we represent the solution as a *single layer boundary integral potential*

$$u(t,x)=S(\partial_t)\varphi(t,x):=\int_0^t\int_\Gamma k(t-\tau,|x-y|)\varphi(\tau,y)d\Gamma_y d\tau, \tag{1.2}$$

where $\varphi\colon [0,T]\times\Gamma\to\mathbb{R}$ is an unknown density and k is the fundamental solution which depends on the spatial dimension

$$k(t,r)=\begin{cases}\dfrac{H(t-r)}{2\pi\sqrt{t^2-r^2}}, & d=2,\\[2mm] \dfrac{\delta(t-r)}{4\pi r}, & d=3,\end{cases} \tag{1.3}$$

where $H(\cdot)$ is the Heaviside function and $\delta(\cdot)$ the Dirac delta distribution.

Taking the trace onto Γ of (1.2) we obtain *the boundary integral equation* for the unknown density φ: Find φ such that

$$V(\partial_t)\varphi(t,x):=\int_0^t\int_\Gamma k(t-\tau,|x-y|)\varphi(\tau,y)d\Gamma_y d\tau=g(t,x)\quad (t,x)\in[0,T]\times\Gamma. \tag{1.4}$$

Once the density φ is obtained, the solution u can be recovered from (1.2). All the numerical methods in this paper will be based on this boundary formulation of the scattering problem.

Note that we have implicitly defined two operators above. The single layer potential $S(\partial_t)\cdot$ and single layer boundary integral operator $V(\partial_t)\cdot$. The motivation for the notation and the mapping properties are described in [10,29] and will be briefly explained in the next section.