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Capturing Vertical Information in Radially Symmetric Flow Using Hyperbolic Shallow Water Moment Equations

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Abstract. Models for shallow water flow often assume that the lateral velocity is constant over the water height. The recently derived shallow water moment equations are an extension of these standard shallow water equations. The extended models allow for a vertically changing velocity profile, resulting in more accuracy when the velocity varies considerably over the height of the fluid. Unfortunately, already the one-dimensional models lack global hyperbolicity, an important property of partial differential equations that ensures that disturbances have a finite propagation speed.

In this paper, cylindrical shallow water moment equations are formulated by starting from the cylindrical incompressible Navier-Stokes equations. We formulate two-dimensional axisymmetric Shallow Water Moment Equations by imposing axisymmetry in the cylindrical model. The loss of hyperbolicity is analyzed and a hyperbolic axisymmetric moment model is then derived by modifying the system matrix in analogy to the one-dimensional case, for which the hyperbolicity problem has already been observed and overcome. Numerical simulations with both discontinuous and continuous initial data in a cylindrical domain are performed using a finite volume scheme tailored to the cylindrical mesh. The newly derived hyperbolic model is clearly beneficial as it gives more stable solutions and still converges to the reference solution when increasing the number of moments.

AMS subject classifications: 35L60, 35L65, 35Q35, 76B07

Key words: First-order nonlinear hyperbolic equations, hyperbolic conservation laws, PDEs in connection with fluid mechanics, radially symmetric free-surface flow.

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1 Introduction

The Shallow Water Equations (SWE) are a set of partial differential equations that describe fluid flows for which the horizontal length scale is much larger than the vertical length scale. Applications can be found in a wide range of scientific fields, such as weather forecasting [25] and free-surface flows like tsunami modelling [9]. A crucial feature of the SWE is that the lateral velocity field is constant over the vertical position variable. This is a severe simplification and renders the SWE inaccurate in applications such as dam-break floods [18] and tsunamis [1]. Consider, for example, small velocities at the bottom of a river, because of the interaction with the bottom friction. Another example is a large velocity just below the surface due to a strong wind blowing above the free surface.

These applications clearly show that a more flexible system of equations is needed for the modeling of flows with complex motion. For this reason, the so-called *Shallow Water Moment Equations (SWME)* were derived in [16]. These equations allow vertical variability in the lateral velocities. The SWME are obtained using the method of moments, which includes an expansion of the lateral velocity in a polynomial basis and a subsequent Galerkin projection to obtain evolution equations for the expansion coefficients. The new system of equations proved to be more accurate than the SWE in numerical simulations [16]. The model was recently extended to the non-hydrostatic case in [20] and generalized to include multi-layer models [7], similar to [6].

When modeling flows in rivers or oceans, complex propagation speeds are nonphysical, because waves propagate with real and finite propagation speeds. Unfortunately, already the one-dimensional SWME are not globally hyperbolic [14] leading to propagation speeds with non-zero imaginary part. The loss of hyperbolicity can lead to instabilities in numerical test cases [14, 23]. The lack of hyperbolicity motivated the derivation of a hyperbolic regularization, the so-called *Hyperbolic Shallow Water Moment Equations (HSWME)* [14], by modifying the system matrix based on similar approaches from kinetic theory [5, 11, 15], thus guaranteeing global hyperbolicity. The hyperbolicity of the HSWME was proved in one spatial dimension and numerical simulations of the one-dimensional HSWME yielded accurate results [14]. This also allowed for a deeper analysis in [12] and [13].

The goal of this paper is to formulate, analyze and simulate moment equations for shallow flows in a cylindrical coordinate system as the necessary step towards a future two-dimensional hyperbolic model. The SWE formulated in cylindrical coordinates are widely used. The reason for this is that for many classical applications such as tsunamis [4, 9] and tropical cyclones [8], a system expressed in cylindrical coordinates is more appropriate. Starting from the cylindrical SWE, an axisymmetric system that models the evolution of flow that only varies in the radial direction can be obtained by setting the angular derivatives to zero. In this way, axisymmetric SWE are classically obtained [19, 21]. We show that the extension to axisymmetric SWME suffers from the same loss of hyperbolicity that is observed in the one-dimensional case [14]. As an exten-