Commun. Comput. Phys.
doi: 10.4208/cicp.OA-2024-0113

Vol. **37**, No. 3, pp. 849-876
March 2025

Accurate Adaptive Deep Learning Method for Solving Elliptic Problems

Jinyong Ying[1], Yaqi Xie[1], Jiao Li[2] and Hongqiao Wang[1,3,*]

[1] *School of Mathematics and Statistics, HNP-LAMA, Central South University, Changsha, Hunan 410083, P.R. China.*
[2] *School of Mathematics and Statistics, Hunan Provincial Key Laboratory of Mathematical Modeling and Analysis in Engineering, Changsha University of Science and Technology, Changsha, Hunan 410114, P.R. China.*
[3] *Institute of Mathematics, Henan Academy of Sciences, Zhengzhou 450046, P.R. China.*

Received 14 May 2024; Accepted (in revised version) 24 October 2024

Abstract. Deep learning method is of great importance in solving partial differential equations. In this paper, inspired by the failure-informed idea proposed by Gao *et al.* (SIAM Journal on Scientific Computing 45(4)(2023)) and as an improvement, a new accurate adaptive deep learning method is proposed for solving elliptic problems, including interface problems and convection-dominated problems. Based on the failure probability framework, the piece-wise uniform distribution is used to approximate the optimal proposal distribution and a kernel-based method is proposed for efficient sampling. Together with the improved Levenberg-Marquardt optimization method, the proposed adaptive deep learning method shows great potential in improving solution accuracy. Numerical tests on the elliptic problems without interface conditions, on one elliptic interface problem, and on the convection-dominated problems demonstrate the effectiveness of the proposed method, as it reduces the relative errors by a factor varying from 10^2 to 10^4 for different cases.

AMS subject classifications: 64C20, 65N20, 68T07

Key words: Deep learning methods, elliptic problems, adaptive sampling method.

1 Introduction

Solving partial differential equations (PDEs) is important in applications and simulations. Due to powerful approximation abilities of neural networks, deep learning methods gain mathematicians' attention. The method was proposed and has been used since

*Corresponding author. *Email addresses:* hongqiao.wang@csu.edu.cn (H. Wang), ying48882345@gmail.com (J. Ying), 847999667@qq.com (Y. Xie), jiaoli@csust.edu.cn (J. Li)

last century [1,2], and became more and more popular as it being able to solve the Poisson equation using the deep Ritz method [3] and other PDEs (as well as other related inverse problems) using the typical physics-informed neural networks (PINNs) [4]. These two methods are similar to each other, just using different loss functions. More specifically, constructions of loss functions for the deep Ritz method are based on weak formulations of symmetric elliptic problems, while loss functions for the PINNs are based on residual forms of the original PDEs. Due to the requirement of symmetry in construction of the loss function, PINNs are more widely used for solving PDEs, compared to the deep Ritz method. As for elliptic interface problems, these two methods encountered low accuracy issues. To overcome this difficulty, there are mainly two types of improvements. One type of improvements is using the domain decomposition idea. That is, to approximate the solution of the interface problem, two or more neural networks instead of a single network are used to simulate the solution within each subdomain. Here multiple networks are 'connected' by enforcing the interface conditions from different subdomains. Using either the residual-based loss function [5] or the weak type loss function [6], the improved deep learning method is shown to be able to get approximations with improved solution accuracy (relative errors being about 10^{-3}) using the Adam solver. The other type is based on the extension-and-projection technique. That is, by augmenting one more dimensionality in spatial variable [7], one network is used to approximate the solution of the elliptic interface problem. Meanwhile, numerical experiments show that using the Levenberg-Marquardt (LM) method instead of the Adam method, the numerical solution can reach quite high accuracy (relative errors being about 10^{-6} or even better for some cases). Further study [8] shows that the improved deep learning method based on the domain decomposition idea can also reach high solution accuracy, although the use of one network can achieve slightly better results than multiple networks. Besides, for the deep learning method, many improvements with respect to different aspects have been proposed in order to gain more accurate approximations, including improving the topology structure of the neural network [9–11] to enhance its representation ability, embedding the gradient information of PDEs [12], and optimally choosing the weights in loss functions [13].

As for solving elliptic problems with complex solution structures (e.g., solution having a 'peak' point or containing the internal/boundary layer), how to effectively choose the training points is significant since a fixed set of (uniformly chosen) training points may fail to capture the effective solution region [14, 15], which leads to the proposals for adaptive deep learning methods based on different strategies [16–22]. Among these, the residual-based adaptive refinement method [17] is the most commonly-used strategy, which adds the sampling points simply based on predictions of residuals of loss functions. Another interesting one is the so-called deep adaptive sampling (DAS) method [18], where a generative model is used to generate new training points based on the residuals. The last one is just the failure-informed PINNs (FI-PINNs) [16], which introduces a novel failure probability estimation and uses a truncated Gaussian-based approximation to automatically generate new training points. More specifically, the FI-