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A Fast Offline/Online Forward Solver for Stationary Transport Equation with Multiple Inflow Boundary Conditions and Varying Coefficients

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Abstract. It is of great interest to solve the inverse problem of stationary radiative transport equation (RTE) in optical tomography. The standard way is to formulate the inverse problem into an optimization problem, but the bottleneck is that one has to solve the forward problem repeatedly, which is time-consuming. Due to the optical property of biological tissue, in real applications, optical thin and thick regions coexist and are adjacent to each other, and the geometry can be complex. To use coarse meshes and save the computational cost, the forward solver has to be asymptotic preserving across the interface (APAL). In this paper, we propose an offline/online solver for RTE. The cost at the offline stage is comparable to classical methods, while the cost at the online stage is much lower. Two cases are considered. One is to solve the RTE with fixed scattering and absorption cross sections while the boundary conditions vary; the other is when cross sections vary in a small domain and the boundary conditions change many times. The solver can be decomposed into offline/online stages in these two cases. One only needs to calculate the offline stage once and update the online stage when the parameters vary. Our proposed solver is much cheaper when one needs to solve RTE with multiple right-hand sides or when the cross sections vary in a small domain, thus can accelerate the speed of solving inverse RTE problems. We illustrate the online/offline decomposition based on the Tailored Finite Point Method (TFPM), which is APAL on general quadrilateral meshes.

AMS subject classifications: 65N06, 76N20, 35Q70

Key words: Asymptotic preserving, offline/online decomposition, radiative transport equation, boundary/interface layer, tailored finite point method.

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1 Introduction

Optical tomography (OT) is a non-invasive functional imaging of tissue to assess physiological function. It can detect and characterize breast cancer or other soft tissue lesions. In OT, a narrow collimated beam is sent into biological tissues, and the light that propagates through the medium is collected by an array of detectors. The sources and measurement locations are adjusted to be able to recover the material properties [1,2].

The propagation of light in complex media can be described by the following stationary RTE

$$\boldsymbol{u}\cdot\nabla\psi(\boldsymbol{r},\boldsymbol{u})+\big(\sigma_S(\boldsymbol{r})+\sigma_a(\boldsymbol{r})\big)\psi(\boldsymbol{r},\boldsymbol{u})=\sigma_S(\boldsymbol{r})\int_S K(\boldsymbol{u},\boldsymbol{u}')\psi(\boldsymbol{r},\boldsymbol{u}')\mathrm{d}\boldsymbol{u}', \tag{1.1}$$

where $\psi(\boldsymbol{r},\boldsymbol{u})$ represents the photon density at position $\boldsymbol{r}\in\Omega\subset\mathbb{R}^3$ and traveling in direction $\boldsymbol{u}\in S$ with S being an unit ball. $\sigma_S(\boldsymbol{r})$, $\sigma_a(\boldsymbol{r})$ represent respectively the scattering cross section and absorption cross section; $K(\boldsymbol{u},\boldsymbol{u}')$ is the scattering kernel that gives the probability that a particle traveling with direction $\boldsymbol{u}'$ being scattered to direction $\boldsymbol{u}$. Boundary conditions are

$$\psi(\boldsymbol{r},\boldsymbol{u})=\psi_{\Gamma^-}(\boldsymbol{r},\boldsymbol{u}),\quad(\boldsymbol{r},\boldsymbol{u})\in\Gamma^-=\{\boldsymbol{r}\in\Gamma=\partial\Omega,\ \boldsymbol{u}\cdot\boldsymbol{n}_{\boldsymbol{r}}<0\}. \tag{1.2}$$

where $\boldsymbol{n}_{\boldsymbol{r}}$ is the outward normal vector at $\boldsymbol{r}\in\Gamma$. In order to probe the structure of highly scattering media, OT needs to solve the inverse stationary radiative transport equation (RTE), which attracts a lot of attention in the past decade. As pointed in [3], due to recent technical development, a vast number of source-detector pairs can be obtained, and it is of great interest to solve inverse RTE with substantial data sets.

Inverse stationary RTE has been extensively studied both analytically and numerically. The uniqueness and stability results have been analyzed in [4,5]. The measured data is usually a bounded linear functional of ψ, which can be denoted by $\mathfrak{M}\psi$. In order to get $\sigma_S(z)$, $\sigma_a(z)$ from the measured data M, one has to iteratively update $\sigma_S(z)$, $\sigma_a(z)$ in such a way that the forward RTE generates $\mathfrak{M}\psi$ that match M with higher and higher accuracy. More precisely, one has to minimize the following objective function

$$\frac{\alpha}{2}\|\mathfrak{M}\psi-M\|^2+\frac{\beta}{2}\mathfrak{R}(\sigma_a,\sigma_S) \tag{1.3}$$

subject to the constraints (1.1). Here α, β are two tune parameters; $\|\mathfrak{M}\psi-M\|^2$ is to quantify the difference between the model predictions and measurements; $\mathfrak{R}(\sigma_a,\sigma_S)$ is the regularization term. In this paper, we consider only inverse boundary value problems in the sense that the measurements are all taken at the boundary. There are two ways to solve the minimization problem, one is to convert (1.3) into an unconstrained optimization problem, and the other is to solve the constrained optimization problem directly [3]. In the first approach, one often first linearizes the problem around some known background to obtain a linear inverse problem. Then Green's functions that solve adjoint RTEs are needed to obtain the constraints that σ_a, σ_S satisfy. One has to solve the forward and adjoint RTEs many times. For the second approach, forward and minimization