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# Stabilized Continuous Linear Element Method for the Biharmonic Problems

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**Abstract.** In this paper, we introduce a new stabilized continuous linear element method for solving biharmonic problems. Leveraging the gradient recovery operator, we reconstruct the discrete Hessian for piecewise continuous linear functions. By adding a stability term to the discrete bilinear form, we bypass the need for the discrete Poincaré inequality. We employ Nitsche's method for weakly enforcing boundary conditions. We establish well-posedness of the solution and derive optimal error estimates in energy and $L^2$ norms. Numerical results are provided to validate our theoretical findings.

**AMS subject classifications**: 65N30, 65N12, 35J15, 35D35

**Key words**: Biharmonic problems, gradient recovery, superconvergence, linear finite element.

## 1 Introduction

In this paper, we consider the numerical approximation of biharmonic equations. Specifically, we focus on the following model problem:

$$\Delta^2 u = f \quad \text{in } \Omega, \tag{1.1a}$$

$$u = \phi \quad \text{on } \partial\Omega, \tag{1.1b}$$

$$\frac{\partial u}{\partial \mathbf{n}} = \psi \quad \text{on } \partial\Omega. \tag{1.1c}$$

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In this context, $\Omega \subset \mathbb{R}^2$ represents a bounded domain with a Lipschitz continuous boundary. The unit outer normal vector of $\Omega$ is denoted as $\mathbf{n}$, and the functions $f \in L^2(\Omega)$, as well as $\phi$ and $\psi$, are given data.

Biharmonic equations find applications in various engineering and physical domains, such as the bending of thin plates in elasticity [7], the stream function-vorticity formulation of Stokes flow [3], and phase field models [14], among others. Over the past decades, we have witnessed significant advancements in numerical methods for solving biharmonic equations. In the existing literature, finite element methods for these equations can be broadly categorized into three classes: conforming, nonconforming, and mixed finite element methods. Conforming finite element methods for biharmonic equations necessitate $C^1$ continuity. Prominent examples, such as Argyris and Bell elements [6], require a minimum of 18 degrees of freedom per element in 2D. To reduce the degrees of freedom, nonconforming elements like the Morley element [13,15,21]), relax the continuity requirements by enforcing weak continuity at specific points. Mixed finite element methods circumvent the need for $C^1$ continuity by reformulating the biharmonic equations into a pair of second-order equations that can be solved using standard Lagrange elements. However, it is worth noting that this approach has the potential to introduce spurious solutions, especially in cases involving simply supported plate boundary conditions [24]. Furthermore, employing a mixed method necessitates solving a saddle point linear system, a process characterized by increased complexity compared to the symmetric positive definite system obtained through a direct discretization of the fourth-order operator.

In recent years, a novel class of finite element methods has emerged for the numerical solution of fourth-order partial differential equations, known as recovery-based finite element methods [1,4,8,10–12,23]. The primary objective of this methodology is to employ the simplest continuous linear finite element for the direct discretization of biharmonic equations. It is crucial to emphasize that the weak formulation of biharmonic problems involves the second derivative. However, the second derivative of continuous linear finite element functions is undefined, which impedes the direct application of the continuous linear finite element method. To overcome this challenge, a gradient recovery operator originally designed for postprocessing purposes is utilized to smooth the discontinuous gradient of continuous linear functions, enabling further differentiation. Notably, Guo et al. in [4, 10] proposed the adoption of classical gradient recovery methods such as the Weighted Average (WA) method [26] and Polynomial Preserving Recovery (PPR) method [16,17] to construct continuous finite element methods for biharmonic equations. This approach, employing only piecewise polynomials, achieves the same order of convergence as other finite element methods employing piecewise quadratic polynomials. They have established optimal error estimates in energy, $H^1$, and $L^2$ norms.

Although the method introduced in Guo et al. [10] demonstrates commendable numerical performance, it remains afflicted by several unresolved issues. This paper aims to tackle these issues by presenting a novel stabilized continuous finite element method for biharmonic equations. First and foremost, the stability analysis of the continuous fi-