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A Coupled High-Order Continuous and Discontinuous Galerkin Finite Element Scheme for the Davey-Stewartson System

Jinyang Lu[1], Qi Tao[2] and Yan Xu[1,*]

[1] *School of Mathematical Sciences, University of Science and Technology of China, Hefei, Anhui 230026, P.R. China.*
[2] *School of Mathematics, Statistics and Mechanics, Beijing University of Technology, Beijing 100124, P.R. China.*

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Abstract. In this paper, we propose a coupled discontinuous Galerkin (DG) and continuous Galerkin (CG) scheme for solving the nonlinear evolution Davey-Stewartson (DS) system in dimensionless form. The DS system consists of two coupled nonlinear and complex structure partial differential equations. The wave's amplitude in the first equation is solved by the high-efficiency local DG method, and the velocity in the second equation is obtained by a standard CG method. No matching conditions are needed for the two finite element spaces since the normal component of the velocity is continuous across element boundaries. The main strengths of our approach are that we combine the advantage of DG and CG methods, using DG methods handling the nonlinear Schrödinger equation to obtain high parallelizability and high-order formal accuracy, using the continuous finite elements solving the velocity to maintain total energy conservation. We prove the energy-conserving properties of our scheme and error estimates in L^2-norm. However, the non-linearity terms bring a lot of trouble to the proof of error estimates. With the help of energy-conserving properties, we construct a series of energy equations to obtain error estimates. Numerical tests for different types of systems are presented to clarify the effectiveness of numerical methods.

AMS subject classifications: 65M60, 35L70, 35Q55, 65M12

Key words: Davey-Stewartson system, local discontinuous Galerkin method, continuous Galerkin method, error estimates.

1 Introduction

In this paper, we introduce a coupled discontinuous Galerkin (DG) and continuous Galerkin (CG) scheme for the following Davey-Stewartson (DS) system

*Corresponding author. *Email addresses:* lujy@mail.ustc.edu.cn (J. Lu), taoqi@bjut.edu.cn (Q. Tao), yxu@ustc.edu.cn (Y. Xu)

$$i\partial_t H+\lambda\partial_x^2 H+\partial_y^2 H=\nu|H|^2 H+\beta H\partial_x\Phi, \tag{1.1a}$$

$$\partial_x^2\Phi+\alpha\partial_y^2\Phi=\chi\partial_x(|H|^2), \tag{1.1b}$$

with periodic boundary condition for H and Dirichlet boundary condition for Φ. Here, H means the amplitude of the wave which is a complex-valued function, and Φ is the real velocity potential of the wave movement. The five parameters λ, ν, β, α and χ are all real numbers that can possess both signs.

The DS system is a kind of nonlinear evolution equation, which originates from fluid mechanics. Davey and Stewartson [1] first introduced this form to describe the evolution in the finite depth of weakly nonlinear water waves in one direction, but their analysis did not take into account the influence of surface tension. This effect was later included by Djordjević and Redekopp [2] who indicated that the parameter α can become negative when surface tension effects are significant. This system also describes short-wave-long-wave resonances and other patterns of propagating waves [3, 4]. In terms of (1.1), the focusing and defocusing case are determined by the parameter ν. According to the respective sign of λ and α, the DS system can be divided into four types [5]:

- Elliptic-Elliptic (EE): $\lambda>0$, $\alpha>0$,
- Hyperbolic-Hyperbolic (HH): $\lambda<0$, $\alpha<0$,
- Elliptic-Hyperbolic (EH): $\lambda>0$, $\alpha<0$,
- Hyperbolic-Elliptic (HE): $\lambda<0$, $\alpha>0$,

or simply DSI system and DSII system, the so-called DSI ($\lambda=1$, $\alpha=-1$) and DSII ($\lambda=-1$, $\alpha=1$) are very special cases of elliptic-hyperbolic and hyperbolic-elliptic DS system respectively, they are always integrable [6]. Usually, the DS system can be regarded as a natural generalization of the cubic nonlinear Schrödinger (NLS) equation

$$i\partial_t H+\Delta H=\nu|H|^2 H. \tag{1.2}$$

Furthermore, the DS system can also be supposed to simplify the Zakharov-Rubenchik and Benney-Roskes system, which serves as an "universal" model for describing long and short waves' interaction. In recent years, the DS system has also appeared in many physics studies, such as plasma physics [7], ferromagnetism [8], nonlinear optics, and so on. We assume that the solution (H,Φ) of system (1.1) is not only smooth enough but also attenuates at infinity. Then the following global quantities E_1, E_2 are constants of motion [5],

$$E_1(H)=\int_\Omega |H|^2\,dxdy, \tag{1.3}$$

$$E_2(H,\Phi)=\int_\Omega \lambda|H_x|^2+|H_y|^2+\frac{1}{2}\Big(\nu|H|^4+\frac{\beta}{\chi}\big((\partial_x\Phi)^2+\alpha(\partial_y\Phi)^2\big)\Big)\,dxdy. \tag{1.4}$$