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# Discontinuous Galerkin Method for the Coupled Dual-Porosity-Navier–Stokes Model

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**Abstract.** In this paper, we extend the weighted discontinuous Galerkin finite element method (WDG) on polygonal grids for solving the dual-porosity-Navier–Stokes model. The Navier–Stokes model describes the free flow in conduits, while the dual-porosity model describes the fluid flow in a medium composed of matrix and microfractures. These two models are coupled through four physically meaningful interface conditions. We obtain the existence and local uniqueness of the solution, as well as the optimal error estimate, under appropriate small data conditions that maintain physical properties. Through numerical experiments, the advantages of the numerical method are verified, such as the optimal convergence rate of the numerical solution to different mesh types and numerical schemes, the performance of the classical upwind scheme combined with the Picard iteration method in handling small viscosity problems, the flow around a horizontal production wellbore with open-hole completion, the different application simulation of multistage hydraulic fractured horizontal wellbore with cased hole completion, as well as the simulation of fluid flow characteristics around macro-fractures.

**AMS subject classifications**: 65N30, 65N15, 65N12

**Key words**: Dual porosity model, Navier–Stokes equations, Beavers–Joseph–Saffman interface condition, weighted discontinuous Galerkin methods, error estimate.

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# 1 Introduction

The flow of incompressible fluid through porous media is widely used in various branches of science and engineering, such as groundwater flow systems [25, 57, 61], industrial filtration [50], and research on geothermal energy [34]. Consequently, many researchers have conducted extensive research into the percolation mechanism in porous media and developed numerous mathematical models to describe the porous flow process, which has advanced the development of science and industry. Among them, the Stokes–Darcy model is the most popular. The Stokes equation is used to describe the flow of fluid in the free fluid region, and the Darcy equation is used to describe the flow of fluid in the porous media region. The fluid interacts between the two regions through appropriate physical conditions.

In recent years, a lot of work has been done on the numerical analysis of the Stokes–Darcy system, including finite element methods [62, 66], domain decomposition methods [23, 24, 36, 51, 97], Lagrange multiplier methods [9, 41, 42, 60], multigrid methods [6, 77], discontinuous Galerkin methods [27, 84, 107, 108], conforming finite volume element method [67], discontinuous finite volume element method [65, 98], mortar finite element methods [13, 14, 38, 39, 48], least square methods [53, 79], partitioned time stepping methods [78, 90, 91], and boundary integral method [17, 95], hybridizable discontinuous Galerkin methods [43], weak Galerkin methods [29, 64, 68], hybrid high order methods [106], and virtual element methods [71].

Up to now, in the coupling of free flow and porous media flow, mathematical modeling and numerical analysis methods have become increasingly abundant. However, most studies are based on homogeneous porous media. In fact, from the perspective of hydrogeology, fractures and discontinuities are one of the most important geological structures. A fracture is typically characterized by a small aperture compared with both its length and the size of the domain and a different porous structure than that of the surrounding medium. The task of effectively modeling the interaction between the system of fractures and the porous matrix is particularly challenging. Depending on the spatial scale under consideration, fractures can be incorporated into such models in two main ways [8]. On the one hand, at small scales, when the number of micro-fractures is large, making it difficult to determine their specific locations, it is very appropriate to use the so-called dual-porosity models [30, 58, 59, 82] to characterize them. In this case, the network of fractures and the bulk or porous matrix are two interacting continua related by a transfer function. On the other hand, at large scales, based on geometrical model reduction techniques, fractures are represented as $d-1$-dimensional interfaces immersed in an $d$-dimensional porous matrix. A coupled bulk-fracture model can handle such a problem as proposed by [73], in which the fracture is supposed to have low thickness with respect to the width of the porous medium. In this case, fractures can behave as either preferential flow paths or geological barriers, depending on the permeability contrast between the porous matrix and the fractures.

The Darcy model widely used in the traditional Stokes–Darcy model generally only