Commun. Comput. Phys.
doi: 10.4208/cicp.OA-2023-0214

Vol. **36**, No. 1, pp. 248-273
July 2024

A Moving Mesh Finite Element Method for Bernoulli Free Boundary Problems

Jinye Shen[1,*], Heng Dai[1] and Weizhang Huang[2]

[1] *School of Mathematics, Southwestern University of Finance and Economics, Chengdu 611130, P.R. China.*
[2] *Department of Mathematics, University of Kansas, Lawrence, Kansas, USA.*

Received 10 August 2023; Accepted (in revised version) 13 January 2024

Abstract. A moving mesh finite element method is studied for the numerical solution of Bernoulli free boundary problems. The method is based on the pseudo-transient continuation with which a moving boundary problem is constructed and its steady-state solution is taken as the solution of the underlying Bernoulli free boundary problem. The moving boundary problem is solved in a split manner at each time step: the moving boundary is updated with the Euler scheme, the interior mesh points are moved using a moving mesh method, and the corresponding initial-boundary value problem is solved using the linear finite element method. The method can take full advantages of both the pseudo-transient continuation and the moving mesh method. Particularly, it is able to move the mesh, free of tangling, to fit the varying domain for a variety of geometries no matter if they are convex or concave. Moreover, it is convergent towards steady state for a broad class of free boundary problems and initial guesses of the free boundary. Numerical examples for Bernoulli free boundary problems with constant and non-constant Bernoulli conditions and for nonlinear free boundary problems are presented to demonstrate the accuracy and robustness of the method and its ability to deal with various geometries and nonlinearities.

AMS subject classifications: 65M60, 65M50, 35R35, 35R37

Key words: Free boundary problem, moving boundary problem, moving mesh, finite element, pseudo-transient continuation.

1 Introduction

Bernoulli free boundary problems (FBPs) arise in ideal fluid dynamics, optimal insulation, and electro chemistry [17] and serve as a prototype of stationary FBPs. They have

*Corresponding author. *Email addresses:* `jyshen@swufe.edu.cn` (J. Shen), `dh20230810@163.com` (H. Dai), `whuang@ku.edu` (W. Huang)

been extensively studied theoretically and numerically; e.g., see [2,7,9,10,12,15,17,36,42]. To be specific, we consider here a typical Bernoulli FBP

$$\begin{cases} -\Delta u = 0, & \text{in } \Omega, \\ u = 1, & \text{on } \Gamma_1, \\ u = 0, & \text{on } \Gamma_2, \\ -\frac{\partial u}{\partial n} = \lambda, & \text{on } \Gamma_2, \end{cases} \tag{1.1}$$

where Ω is a connected domain in $\mathbb{R}^2$ (see Fig. 1), λ is a positive constant, $\Gamma_1 \cup \Gamma_2 = \partial\Omega$, Γ_1 is given and fixed, and Γ_2 is unknown a priori and part of the solution. We emphasize that the numerical method studied in this work can be applied to more general FBPs without major modifications, and several such examples are presented in Section 5.

The Neumann boundary condition in (1.1) is called the Bernoulli condition. This condition can be shown to be equivalent to $|\nabla u| = \lambda$ (with the help of the Dirichlet boundary condition on Γ_2). Moreover, the problem is called an exterior (or interior) Bernoulli problem when Γ_2 is exterior (or interior) to Γ_1 (cf. Fig. 1). It is known [2,5,17] that an exterior Bernoulli problem has a solution for any $\lambda > 0$ and such a solution is unique and elliptic when the domain enclosed by Γ_1 is convex. Loosely speaking, a solution is said to be elliptic (or hyperbolic) if Γ_2 is getting closer to (or moving away from) Γ_1 as λ increases. On the other hand, an interior Bernoulli problem has a solution only for λ large enough and such solutions are not unique in general. Both elliptic and hyperbolic solutions can co-exist for the same value of λ for interior problems.

While the differential equation and boundary conditions are linear, the problem (1.1) is actually highly nonlinear due to the coupling between u and Ω. A number of numerical methods have been developed for solving Bernoulli FBPs; e.g., see a summary of early works for general FBPs [12, Chapter 8], the explicit and implicit Neumann methods [17], a combined level set and boundary element method [31], shape-optimization-based methods [14, 15, 21, 36], the cut finite element method [9], the quasi-Monte Carlo method [7], the comoving mesh method [40], and the singular boundary method [11]. A common theme among those methods is trial free boundary and thus iterating between the update of the free boundary and the solution of the corresponding boundary value problem. Challenges for this approach include how to choose the initial guess for Γ_2 to make the iteration convergent and to re-generate or deform the mesh to fit the varying domain.

In this work we shall present a moving mesh finite element method for the numerical solution of Bernoulli FBPs. The method is based on the pseudo-transient continuation (e.g., see Fletcher [16, Section 6.4]) with which we construct an equivalent time-dependent problem (a moving boundary problem or an MBP), march it until the steady state is reached, and take the steady-state solution as the solution of Bernoulli FBP (1.1). The pseudo-transient continuation is widely used for difficult nonlinear problems in science and engineering because it can be made convergent for a large class of initial solutions. Another advantage of using the pseudo-transient continuation is that the corre-