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High Order Bound- and Positivity-Preserving Finite Difference Affine-Invariant AWENO Scheme for the Five-Equation Model of Two-Medium Flows

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Abstract. Numerical study on compressible two-medium flows has been a hot issue in recent decades. In this study, we design quasi-conservative finite difference alternative weighted essentially non-oscillatory (AWENO) schemes up to the ninth order for the five-equation model with the stiffened gas equation of state. We propose uniformly high-order flux-based bound- and positivity-preserving (BP-P) limiters for the AWENO schemes while preserving the equilibrium solutions simultaneously. Though the BP-P limiters are used, the numerical solutions have the tendency to generate oscillations especially near strong shock and/or rarefaction waves, due to the sudden drastic scale transition of the density, pressure, etc. To resolve fine structures and the transition of different scales, the latest affine-invariant WENO (Ai-WENO) interpolation is adopted and generalized up to the ninth order. In addition, we will systematically derive CFL conditions when the Lax-Friedrichs numerical flux is applied. Moreover, we show the potential of the BP-P limiters for a variant of the five-equation model, usually suggested in finite volume and discontinuous Galerkin methods. For illustration purposes, we adopt the AWENO schemes and derive the corresponding CFL conditions. A variety of one- and two-dimensional test problems illustrate the high order of accuracy, effectiveness, and robustness of the proposed BP-P Ai-AWENO schemes.

AMS subject classifications: 76-XX

Key words: Compressible two-medium flow, AWENO scheme, affine-invariant WENO interpolation, bound- and positivity-preserving limiters.

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1 Introduction

Numerical study on compressible two-medium flows with immiscible interfaces, such as gas-gas and/or liquid-gas interactions, has drawn much interest among researchers in recent decades. In this paper, we are interested in the five-equation model [1], in which each physical quantity is uniquely identified under the isobaric assumption. The five-equation model has the potential to describe fluid media associated with a variety of equations of state (EOS), such as ideal gas EOS, stiffened gas EOS, van der Waals gas EOS, Mie-Grüneisen material EOS, and so on. However, numerical inaccuracies and oscillations are inevitable near the material interface since the equation of state would switch between different media across the material interface. A large number of studies show that the E-property, i.e. the velocity and pressure remain unchanged when they are initially constants, should be well preserved by the numerical scheme so that numerical oscillations can be effectively suppressed near the material interface; consult [17–19,21,22,26,27,32–36,41,43,58].

To design numerical discretizations, high order methods are usually recommended. Weighted essentially non-oscillatory (WENO) schemes play important roles in high order methods. The WENO scheme was originally proposed in [31]. Jiang and Shu generalized the WENO scheme by introducing the so-called non-linear WENO-JS weights and gave an effective implementation strategy [24]. In [2], Borgers *et al.* improved the WENO scheme by introducing the non-linear WENO-Z weights. The non-linearity adapts either to an optimal order approximation if the function is smooth or to a stencil biasing that minimizes the contribution from one or more polynomials on the discontinuous stencils and provides the necessary dissipation needed for shock capturing in an essentially non-oscillatory (ENO) manner, consult [39] and references therein for details and history of the WENO schemes.

Several types of low-dissipation WENO schemes were proposed in the literature. Qiu and Shu designed the original Hermite WENO (HWENO) scheme and discussed its application as limiters for the discontinuous Galerkin (DG) method in [38]. Comparing with the regular WENO schemes, one major advantage of HWENO schemes is their relatively compact stencil. Further improvements and applications of HWENO schemes can be seen in [28, 30, 64]. Fu *et al.* proposed the targeted essentially non-oscillatory (TENO) schemes [14] that offer low dissipation solutions for waves with wave number from low to intermediate range, sharp shock-capturing capacity and good robustness. Takagi *et al.* took it a step further by proposing a high order shock-capturing scheme [42] that combines the TENO scheme for smooth regions with the non-polynomial Tangent of Hyperbola for Interface Capturing (THINC) reconstruction for non-smooth discontinuities. Zhu *et al.* proposed a family of the low-dissipation multi-resolution WENO (MR-WENO) schemes. MR-WENO schemes are capable of handling flow discontinuities and accurately resolving a broad range of length scales; see [51, 65] and references therein. Due to the difference of sub-stencil selection, design of linear and non-linear weights, the methods mentioned above have their own application scenario. For comprehensive