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Relaxation Exponential Runge–Kutta Methods and Their Applications to Semilinear Dissipative/Conservative Systems

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Abstract. This paper presents a family of novel relaxation exponential Runge–Kutta methods for semilinear partial differential equations with dissipative/conservative energy. The novel methods are developed by using the relaxation idea and adding a well-designed governing equation to explicit exponential Runge–Kutta methods. It is shown that the proposed methods can be of high-order accuracy and energy-stable/conserving with mild time step restrictions. In contrast, the previous explicit exponential-type methods are not energy-conserving. Several numerical experiments on KdV equations, Schrödinger equations and Navier–Stokes equations are carried out to illustrate the effectiveness and high efficiency of the methods.

AMS subject classifications: 65L04, 65L06, 65M12
Key words: Explicit exponential Runge–Kutta methods, relaxation technique, high-order accuracy, structure-preserving property.

1 Introduction

Consider the following semilinear partial differential equations (PDEs)

$$\partial_t \phi = \mathcal{L}\phi + \mathcal{N}(\phi), \quad \mathbf{x} \in \Omega \subset \mathbb{R}^d, \quad 0 < t \leq T, \tag{1.1}$$

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where $\mathcal{L}$ is a linear operator and $\mathcal{N}$ a nonlinear operator. It is further assumed that problems (1.1) possess smooth energy functionals $\mathcal{E}(\phi)$, satisfying

$$\frac{d}{dt}\mathcal{E}(\phi)=\int_{\Omega}\frac{\delta\mathcal{E}}{\delta\phi}\phi_t d\mathbf{x}=\int_{\Omega}\frac{\delta\mathcal{E}}{\delta\phi}(\mathcal{L}\phi+\mathcal{N}(\phi))d\mathbf{x}\leq 0, \tag{1.2}$$

and we call them dissipative problems. Specially, the dissipative problems reduce to conservative ones if

$$\frac{d}{dt}\mathcal{E}(\phi)=\int_{\Omega}\frac{\delta\mathcal{E}}{\delta\phi}\phi_t d\mathbf{x}=\int_{\Omega}\frac{\delta\mathcal{E}}{\delta\phi}(\mathcal{L}\phi+\mathcal{N}(\phi))d\mathbf{x}=0. \tag{1.3}$$

Typical models satisfying (1.2) or (1.3) include Korteweg-de Vries (KdV) equation [1–3], nonlinear Schrödinger equation [4–8], incompressible Navier–Stokes equation [9–13] and so on [14–20]. When numerically solving (1.1), it is highly desirable that the numerical methods can preserve the energy dissipation/conservation law at the discrete level.

It has been one of the hot spots to develop energy-stable/conserving methods for problems (1.1). A crucial factor in constructing structure-preserving methods is the choice of time discretization, which can be algebraically stable Runge–Kutta methods [15,21–24], some projection methods [25–29], discrete gradient methods [30, 31], averaged vector field methods [32], continuous-stage Runge–Kutta methods [33] and so on [34–39]. Besides, the convex splitting approach [40–42] and the stabilized approach [43–45] are widely used to develop energy-stable schemes for dissipative problems. The schemes based on the two approaches are usually first- or second-order accurate in temporal direction. After that, the invariant energy quadratization approach [46, 47] and the scalar auxiliary variable approach [18, 19] were proposed to obtain some linearly implicit energy-stable/conserving methods, which have successful applications in [48–53]. These linearized methods satisfy the dissipation/conservation law with respect to some modified energy. Recently, the relaxation idea was applied to develop some energy-stable/conserving methods, see e.g., [54–59]. These relaxation methods are shown to be of high-order accuracy in temporal direction.

Note that problems (1.1) usually consist of linear stiff term and nonlinear mild-/non-stiff term, it is interesting to integrate the linear stiff part exactly and the nonlinearity explicitly, which leads to explicit exponential-type methods. Up to now, explicit exponential-type methods that were proved to be energy-stable are only first- and second-order accurate, see e.g., [12,60–64]. There is no energy-stable explicit exponential-type method of arbitrarily high order. Especially, we have not found any energy-conserving explicit exponential-type methods.

In this paper, we aim to construct high-order structure-preserving exponential-type methods for the time integration of (1.1). Firstly, the explicit exponential Runge–Kutta (ERK) method [65] is applied to obtain the internal stages. Secondly, we utilize the relaxation technique and introduce a relaxation parameter into the step update, which is determined by a well-designed governing equation with respect to the energy. This yields