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# Efficient Implementation of 3D FEM for Nonlocal Poisson Problem with Different Ball Approximation Strategies

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**Abstract.** Nonlocality brings many challenges to the implementation of finite element methods (FEM) for nonlocal problems, such as a large number of neighborhood query operations being invoked on the meshes. Besides, the interactions are usually limited to Euclidean balls, so direct numerical integrals often introduce numerical errors. The issues of interactions between the ball and finite elements have to be carefully dealt with, such as using ball approximation strategies. In this paper, an efficient representation and construction methods for approximate balls are presented based on the combinatorial map, and an efficient parallel algorithm is also designed for the assembly of nonlocal linear systems. Specifically, a new ball approximation method based on Monte Carlo integrals, i.e., the fullcaps method, is also proposed to compute numerical integrals over the intersection region of an element with the ball.

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**Key words**: Nonlocal problem, finite element method, combinatorial map, approximate ball, Monte Carlo integration, parallel computing.

## 1 Introduction

The nonlocal operator has been applied in various fields [1–3]. Due to its wide applications, many numerical algorithms have been developed for solving nonlocal problems effectively, including finite difference method [4,5], finite element method (FEM) [6,7] and collocation method [8,9]. Among these methods, FEM is a powerful discretization technique that uses general unstructured grids to approximate the solutions of many partial

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differential equations (PDEs), and has been studied to solve nonlocal problems [10–12]. But, nonlocality brings some new difficulties in FEM implementations, especially for three-dimensional (3D) cases.

A comprehensive description of the computational challenges can be found in [7] for the FEM implementation of nonlocal problems. Specifically, the integration is more complicated for nonlocal problems, and additional errors will be introduced if classical quadrature rules are directly used to compute the integrals. To avoid this issue, a new method is proposed in [7] by introducing the concept of approximate balls. Pasetto et al. [13] compute the inner integration by using quadrature points distributed over the full ball. A technique that allows direct computation of the inner integral over each element is presented in [14] by smoothing the kernel function.

While the finite element solutions of nonlocal problems with uniform meshes or quasi-uniform meshes [15–21] are well studied and fast stiffness matrix assembly [16, 22–24] is also well developed, there are few works on the implementation of FEM with unstructured meshes. One of the main reasons is that the solutions to nonlocal problems require a large number of element calls and queries to the unstructured meshes. In [7] and [25], the ball approximation strategies are introduced to deal with integrals over the interaction domain of interaction ball and element more precisely, which also increases the difficulty to obtain an approximated ball on unstructured meshes.

The aim of this work is to extend the concept of ball approximation strategies proposed in [7] on 2D unstructured meshes to develop an efficient implementation of 3D FEM for nonlocal Poisson problem. To do so, we first introduce a new data structure called combinatorial maps [26, 27] to handle operations on meshes, including queries and modifying unstructured meshes dynamically, which has been successfully applied in the field of computer graphics [28]. After that, we discuss the implementation issues of FEM for solving nonlocal problems in high dimensions using combinatorial map in detail, including efficient neighborhood queries, ball approximation strategies and the fast matrix assembly needed by nonlocal problems' solution.

The organization of this paper is given as follows. In Section 2, the definitions and notations of nonlocal problems and the weak form of the nonlocal Poisson problems are reviewed. In Section 3, we discuss in detail the definitions of the ball approximation strategies in nD. The estimates of geometric errors of these ball approximation strategies in nD are also presented. In Section 4, we discuss the implementation of FEM for nonlocal problems based on combinatorial map data structure. We then design some algorithms for constructing the nonlocal approximate ball, such as topological iterators developed based on the combinatorial map. Subsequently, we present a detailed assembly procedure to compute numerical solutions for the nonlocal model. Finally, in Section 5, the 3D numerical result shows the effectiveness and accuracy of our implementation.