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Local Trajectory Variation Exponent (LTVE) for Visualizing Dynamical Systems

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Abstract. The identification and visualization of Lagrangian structures in flows plays a crucial role in the study of dynamic systems and fluid dynamics. The Finite Time Lyapunov Exponent (FTLE) has been widely used for this purpose; however, it only approximates the flow by considering the positions of particles at the initial and final times, ignoring the actual trajectory of the particle. To overcome this limitation, we propose a novel quantity that extends and generalizes the FTLE by incorporating trajectory metrics as a measure of similarity between trajectories. Our proposed method utilizes trajectory metrics to quantify the distance between trajectories, providing a more robust and accurate measure of the LCS. By incorporating trajectory metrics, we can capture the actual path of the particle and account for its behavior over time, resulting in a more comprehensive analysis of the flow. Our approach extends the traditional FTLE approach to include trajectory metrics as a means of capturing the complexity of the flow.

AMS subject classifications: 37M05, 37M10, 37M25, 65L05

Key words: Lagrangian coherent structure, trajectory metric, trajectory analysis, finite time Lyapunov exponent.

1 Introduction

The study of coherent structures has been a fundamental area of research in dynamic systems and computational fluid dynamics. Among the various types of coherent structures, Lagrangian coherent structures (LCS) are particularly noteworthy. LCS focuses on identifying regions within a flow that display strong attraction and repulsion of particles over a specific time span. It serves as a valuable tool for analyzing, visualizing, and extracting information from complex dynamical systems. The primary objective is to delineate

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surfaces of trajectories and separate regions in the phase space that exert significant influence on neighboring trajectories during a defined time interval [15]. LCS has found applications in diverse fields, including oceanography [18, 31], meteorology [29], flight mechanics [4, 33], gravity waves [35], and bio-inspired fluid flows [11, 23, 24]. Although the underlying physical intuition behind LCS is straightforward, an accurate and formal mathematical definition capturing this behavior is still a subject of ongoing development. Numerous studies have attempted to establish robust frameworks for identifying LCS structures [14, 31], but these frameworks often face computational challenges or suffer from certain limitations that affect their precision.

The finite-time Lyapunov exponent (FTLE) is a widely used measure for locating Lagrangian coherent structures (LCS) [12, 13, 16, 19, 31]. It quantifies the rate of change in distance between neighboring particles over a finite time interval, considering an infinitesimal perturbation in the initial position. Computing the FTLE field involves several steps. Firstly, the flow map is computed, which establishes the connection between the initial and arrival positions of particles along the characteristic line. The FTLE is then defined based on the largest eigenvalue of the deformation matrix derived from the Jacobian of the flow map. Various Eulerian approaches have been developed to numerically compute the FTLE on a fixed Cartesian mesh in recent studies. For more in-depth discussions on this topic, interested readers can refer to [20–22,27,42–47] and related references.

Another approach to analyzing trajectories involves trajectory analysis, which aims to develop techniques for understanding and classifying trajectory characteristics. While trajectory clustering is not a new topic, previous methods primarily focused on visualizing or constructing dictionaries and summaries to reveal hidden patterns or predict future routes [8–10, 17, 38–40]. These approaches were not specifically designed to discover coherent structures within underlying flows and dynamical systems. However, recent works have explored clustering-based approaches for identifying LCS. Some of these methods involve constructing networks of trajectories and determining edge weights based on trajectory similarity [25,28,30]. In [41], a coherent ergodic partition method was developed to separate trajectories into clusters by integrating and computing long-time averages of functions along the trajectories. This effectively projects high-dimensional data onto a low-dimensional manifold, allowing for partitioning based on these function averages. In [6], a clustering-based approach was proposed to identify coherent flow structures in continuous dynamical systems. This method treats particle trajectories over a finite time interval as high-dimensional data points and clusters them from different initial locations into groups using normalized standard deviation or mean absolute deviation to quantify deformation.

The aim of this paper is to propose a general framework based on the trajectory analysis technique to the study of LCS. Specifically, we seek to develop a new metric for efficiently extracting and visualizing LCS using trajectory analysis. Instead of clustering all provided trajectories into groups, we investigate the variation of the trajectories in a small neighborhood of each trajectory. We define a local quantity that measures the similarity of these trajectories from a small local neighborhood based on various possi-