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Relaxation Schemes for Entropy Dissipative Systems of Viscous Conservation Laws

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Abstract. In this paper, a hyperbolic relaxation model is designed for a class of entropy dissipative systems of viscous conservation laws, such as the 1-D viscous Burgers and 2-D Navier-Stokes equations. An artificial variable is introduced to relax both the convective and viscous fluxes together. Based on the entropy dissipative property of the original system, a dissipation condition is proposed for the resulting relaxation model, and used to prove the entropy inequality of the relaxation model for linear convection-diffusion equations. Lax-Wendroff type second-order finite-volume schemes are developed to discretize the relaxation model. A number of numerical experiments, including viscous compressible flow problems from subsonic to supersonic speeds, are used to validate the relaxation model and evaluate the performance of the current schemes.

AMS subject classifications: 65M08, 76M12, 35Q30, 76N15

Key words: Relaxation method, viscous conservation laws, entropy dissipation, Lax-Wendroff type solver, Navier-Stokes equations, generalized Riemann problem.

1 Introduction

In the past decades, relaxation approaches have gained significant popularity as effective tools in designing numerical schemes for gas dynamics. Nishikawa [33, 34] constructed hyperbolic Navier-Stokes (HNS) models with relaxation for the diffusive flux and investigated numerical methodologies for computing steady-state solutions. This approach was later extended to unsteady problems [27, 29]. Montecinos and Toro [30] proposed relaxation formulations for time-dependent, nonlinear systems of advection–diffusion–reaction

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equations by replacing the spatial gradient in the viscous flux with a new vector of unknowns, and presented locally implicit ADER schemes to solve the resulting stiff systems. Dumbser et al. [14] developed numerical schemes for a unified first order hyperbolic formulation of continuum mechanics, which approximates the Navier-Stokes equations in the stiff relaxation limit.

The above mentioned relaxation models [14, 27, 29, 30, 33, 34] are of the Cattaneo type model [7], which mainly focuses on simplifying the numerical discretization of the second-order diffusion terms in original systems. There exists another relaxation framework for hyperbolic conservation laws, which aims at simplifying the numerical discretization to the nonlinear convective (inviscid) flux. Jin and Xin [20] transformed multidimensional hyperbolic systems of conservation laws into enlarged linear hyperbolic systems with stiff source terms, and proposed relaxation schemes without using nonlinear Riemann solvers. Later, Naldi, Pareschi and Toscani [32] extended this approach to one-dimensional (1-D) systems of nonlinear parabolic equations by replacing both the convective and diffusive fluxes with a new vector of unknowns. Very recently, Wissocq and Abgrall [43,44] proposed a kinetic method to approximate nonlinear advection-diffusion equations, including the compressible Navier-Stokes equations, by considering Jin-Xin type models in the diffusion limit, where the diffusive fluxes are recovered as the first-order term of an asymptotic expansion in a small parameter such that stiff flux terms can be avoided. Here we also mention that the gas-kinetic scheme (GKS) for Navier-Stokes equations by Xu [46], where the governing equation, the kinetic Bhatnagar-Gross-Krook (BGK) model, is a linear hyperbolic equation with relaxation source terms. The relaxation technique is used to developed relaxation exponential Runge-Kutta methods for semilinear equations [24].

We emphasis that the Jin-Xin type relaxation approach in [32] considers only the system of convection-reaction-diffusion equations in the form

$$\partial_t \mathbf{u} + \partial_x \mathbf{F}(\mathbf{u}) = \partial_{xx} \mathbf{P}(\mathbf{u}) + \mathbf{S}(\mathbf{u}), \quad x \in \mathbb{R}, \quad t \in \mathbb{R}^+, \tag{1.1}$$

where $\mathbf{u} \in \mathbb{R}^n$, and $\mathbf{F}(\mathbf{u})$, $\mathbf{P}(\mathbf{u})$ and $\mathbf{S}(\mathbf{u})$ are smooth functions of $\mathbf{u}$. However, the above system with diffusion terms in this form excludes many important models for gas dynamics, such as the compressible Navier-Stokes equations with the viscous flux in the form $\mathbf{B}(\mathbf{u})\nabla\mathbf{u}$, where $\mathbf{B}(\mathbf{u})$ stands for the viscous tensor and is neither symmetric nor semi-positive definite (see Appendix A for more details). In addition, there are many studies on the diffusive limit of numerical schemes for hyperbolic systems with stiff relaxation [5,18,31] and transport equations [21,23,36,37]. However, the limiting diffusion models of the these works do not include the compressible Naiver-Stokes equations.

In the spirit of the Jin-Xin model [20] for hyperbolic conservation laws, this paper makes effort to develop hyperbolic relaxation models for a class of viscous conservation laws, including the compressible Navier-Stokes equations. Novelties of this work are highlighted below. We construct a hyperbolic relaxation model for a class of entropy dissipative systems of viscous conservation laws, and propose a dissipation condition by