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Unfitted Spectral Element Method for Interfacial Models

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Abstract. In this paper, we propose an unfitted spectral element method for solving elliptic interface and corresponding eigenvalue problems. The novelty of the proposed method lies in its combination of the spectral accuracy of the spectral element method and the flexibility of the unfitted Nitsche's method. We also use tailored ghost penalty terms to enhance its robustness. We establish optimal hp convergence rates for both elliptic interface problems and interface eigenvalue problems. Additionally, we demonstrate spectral accuracy for model problems in terms of polynomial degree.

AMS subject classifications: 65N30, 65N25, 65N15

Key words: Elliptic interface problem, interface eigenvalue problem, unfitted Nitsche's method, hp estimate, ghost penalty.

1 Introduction

Interface problems arise naturally in various physical systems characterized by different background materials, with extensive applications in diverse fields such as fluid mechanics and materials science. The primary challenge for interface problems is the low regularity of the solution across the interface. In the pioneering investigation of the finite element method for interface problems [2], Babuška established that standard finite element methods can only achieve $\mathcal{O}(h^{1/2})$ accuracy unless the meshes conform precisely to the interface. To date, various methods have been developed to address interface problems. The existing numerical methods can be roughly categorized into two different classes: body-fitted mesh methods and unfitted methods. When the meshes are fitted

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to the interface, optimal a priori error estimates are established [2, 16, 41]. To alleviate the need for body-fitted mesh generation for geometrically complicated interfaces, various unfitted numerical methods have been developed since the seminal work of Peskin on the immersed boundary method [36]. Famous examples include immersed interface methods [29], immersed finite element methods [30,31], ghost fluid method [33], Petrov-Galerkin methods [26, 27], generalized/extended finite element methods [3, 34], and cut finite element methods [11, 23].

The cut finite element method (CutFEM), also known as the unfitted Nitsche's method, was initially introduced by Hansbo et al. [23]. The key idea behind this approach involves employing two distinct sets of basis functions on the interface elements. These sets of basis functions are weakly coupled using Nitsche's methods. Notably, this idea has been generalized to address various model equations, including elastic interface problems [24], Stokes interface problems [25], Maxwell interface problems [32], and biharmonic interface problems [13]. In our recent research [19], we have established superconvergence results for the cut finite element method. Moreover, high-order analogues of this method have been developed [7, 15, 22, 28, 35, 40]. To enhance the robustness of the method in the presence of arbitrarily small intersections between geometric and numerical meshes, innovative techniques such as the ghost penalty method [10, 12] and the cell aggregation technique [6,8] have been proposed. For readers interested in a comprehensive overview of CutFEM, we refer them to the review provided in [11].

The motivation for our paper stems from our recent investigation of unfitted finite element methods for interface eigenvalue problems [20,21]. These types of interface eigenvalue problems have important applications in materials sciences, particularly in band gap computations for photonic/phononic crystals and in edge model computations for topological materials. In the context of eigenvalue problems, as elucidated in [42], higher-order numerical methods, especially spectral methods/spectral element methods, offer significantly more reliable numerical eigenvalues.

Our paper aims to introduce a novel and robust unfitted spectral element method for solving elliptic interface problems and associated interface eigenvalue problems. Unlike previous methodologies, we emphasize the utilization of nodal basis functions derived from the Legendre-Gauss-Lobatto points [39] and the development of hp error estimates. In pursuit of enhanced robustness, we incorporate a ghost penalty stabilization term with parameters tailored to the polynomial order p. Notably, for interface eigenvalue problems, both mass and stiffness terms require the inclusion of the ghost penalty. However, introducing an additional ghost penalty term in the mass term precludes the direct application of the Babuška-Osborne theory. To overcome this challenge, we propose a solution by introducing intermediate interface eigenvalue problems and their corresponding solution operators. Using the intermediate solution operator as a bridge, we decompose the eigenvalue approximation into two components: one that can be rigorously analyzed using the Babuška-Osborne theory and another that can be estimated through the operator super-approximation property.

The rest of the paper is organized as follows: In Section 2, we introduce the equations