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High Order Asymptotic Preserving Well-Balanced Finite Difference WENO Schemes for All-Mach Full Euler Equations with Gravity

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Abstract. In this paper, we propose a high order semi-implicit well-balanced finite difference scheme for all-Mach Euler equations with a gravitational source. We start with the conservative form of full compressible Euler equations including conservation of total energy, for shock capturing in the high Mach regime. For asymptotic preserving in the low Mach regime and to address the difficulty of strong coupling between the stiff gravitational source and conservative variables, we add the evolution equation of the perturbation of potential temperature, which corresponds to weak potential temperature stratification under a hydrostatic background potential temperature. The resulting system is then split into a (non-stiff) nonlinear low dynamic material wave to be treated explicitly, and (stiff) fast acoustic and gravity waves to be treated implicitly. With the aid of explicit time evolution for the perturbation of potential temperature, we design a novel well-balanced finite difference weighted essentially non-oscillatory (WENO) scheme, which can be proven to be both asymptotic preserving and asymptotically accurate in the incompressible limit. Numerical experiments are provided to validate these properties.

AMS subject classifications: 65M06, 65L04, 65M25, 65M12

Key words: Compressible Euler equations, all-Mach numbers, finite difference WENO, asymp-totic preserving, well-balanced, gravity.

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1 Introduction

In this paper, we are interested in the compressible Euler equations with a gravitational source, which widely appear in astrophysics and meteorology

$$\begin{cases} \rho_t+\nabla\cdot(\rho\mathbf{u})=0, \\ (\rho\mathbf{u})_t+\nabla\cdot(\rho\mathbf{u}\otimes\mathbf{u})+\nabla p=-\rho\nabla\Phi, \\ E_t+\nabla\cdot((E+p)\mathbf{u})=-\rho\mathbf{u}\cdot\nabla\Phi, \end{cases} \tag{1.1}$$

where ρ is the density, $\mathbf{u}$ is the velocity, p is the pressure, and $\rho\mathbf{u}$ is the momentum. $\Phi=\Phi(\mathbf{x})$ is the gravitational potential which is assumed to be time-independent. $E=\frac{1}{2}\rho|\mathbf{u}|^2+\rho e$ is the total non-gravitational energy, with e being the specific internal energy. All the variables are defined on a time-space domain $(t,\mathbf{x})\in\mathbb{R}^+\times\Omega$, where $\Omega\subseteq\mathbb{R}^d$ with d being the dimension of $\mathbf{x}$. And the system (1.1) needs to be closed by providing an equation of state (EOS), which is usually given in the form $e=\mathcal{E}(\rho,p)$. For an ideal gas, the EOS can be written as $e=p/((\gamma-1)\rho)$, and the energy E becomes

$$E=\frac{1}{2}\rho|\mathbf{u}|^2+\frac{p}{\gamma-1}, \tag{1.2}$$

with $\gamma>1$ being the ratio of specific heat.

For such a hyperbolic system with a source term, an important feature is that they admit equilibrium state solutions. A simple equilibrium state of (1.1) is the hydrostatic equilibrium state of the form [77]

$$\mathbf{u}=\mathbf{0}, \quad \nabla p=-\rho\nabla\Phi. \tag{1.3}$$

Numerically, it is desirable to develop well-balanced numerical methods, which can exactly preserve such an equilibrium state, so that small perturbations around an equilibrium state can be well captured on relatively coarse mesh sizes. Many high order well-balanced schemes have been studied, including finite difference schemes [51,77], finite volume schemes [6–8,21,27,34,40,44,48,70], and discontinuous Galerkin finite element methods [22,49,50,72]. Well-balanced schemes are also very attractive for shallow water equations with source terms, see [2,9,47,57,74–76], review papers [46,73], and many references therein. These schemes are mostly based on explicit time evolutions for compressible flows in the high Mach regime.

However, for many applications in atmospheric flows, sound waves are considered unimportant. Explicit schemes for compressible flows mentioned above encounter numerical difficulties, due to the fast characteristic time scale caused by sound waves. Sound proof models, which try to eliminate sound modes, while maintaining other important advection and internal gravity wave modes, are developed, such as anelastic models [3,30,55,59] and pseudo-incompressible models [28,29]. We refer to [43] for a