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An Efficient Numerical Algorithm for Solving Coupled Time-Dependent Ginzburg-Landau Equation for Superconductivity and Elasticity

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Abstract. A decoupled finite element algorithm is developed for simulating the vortex dynamics on an elastic superconductor which couples the time-dependent Ginzburg-Landau equation with the complex-valued superconducting order parameter and the vector-valued magnetic potential, and the elasticity equation. We present an iterative algorithm for the decoupled system arising from the time and spatial discretization using a combination of preconditioner, algebraic multigrid method (AMG) and preconditioned conjugate gradient method (PCG). The iterative algorithm allows us to perform large-scale three-dimensional simulations of mesoscale pattern formation during superconducting phase transitions with arbitrary elastic boundary conditions. The performance and efficiency of the algorithm are numerically verified by several benchmark problems, exhibiting up to two orders of magnitude improvement depending on the scale of discrete system compared to the exact solver.

AMS subject classifications: 65F08, 65M55, 65M60
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1 Introduction

The time-dependent Ginzburg-Landau (TDGL) equations for superconductivity under the temporal gauge have been of great interest to the physical science and engineering

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community [1, 10, 15, 25, 27, 30, 31], and as a result a number of numerical methods for solving the TDGL equations have been proposed and studied in literature, see for example [6, 7, 11] and the references therein. The global well-posedness of the Ginzburg-Landau equations under Lorentz gauge has been analyzed in [20] for superconductors which are not necessarily convex. Several mixed finite element methods have been employed for the Lorentz gauge to eliminate the spurious vortex patterns by conventional methods, see [6, 9, 12, 13, 19, 21].

A number of recent studies have shown increasing interest in the effect that the strain state of the underlying lattice may have on the superconducting transition. For small but significant values of strains which are achievable through epitaxial substrate strain in thin films, previous studies have shown that the relationship between T_c and strain is approximately linear for many materials, while in others the effect is much greater due to being in proximity to electronic or structural transitions [8, 28, 29]. In order to explore these effects, as well as the effects of strain on the critical magnetic field, a model which couples the material strain with the superconducting order parameter is required. In [23], the interaction of the superconducting condensate with deformations of the crystal lattice is formulated assuming the electrostatic potential to be of Bernoulli type while taking into account the effect of strain on material parameters. A phenomenological theory of strain and stress effects in Ginzburg-Landau superconductors was discussed in [18].

Here, we focus on developing efficient numerical solutions to the TDGL equations coupled with the elasticity equation. To our knowledge, this is the first model designed for large-scale simulations of the TDGL equations coupled with the elasticity equation in thin-films. In addition, we demonstrate the efficiency of our proposed scheme and solvers by performing numerical experiments in three dimensions (3D) and comparing the efficiency with a number of algorithms. We solve the system using a direct method and show that even for small systems, the computational cost becomes quickly intractable. Additionally, when compared with the algebraic multigrid (AMG) [3, 33] method alone, our solver shows significantly better performance.

2 Model description

In this section, we describe the modified Time-Dependent Ginzburg-Landau (TDGL) model along with the mechanical equilibrium equation and linear elasticity. In the TDGL model, the superconducting phase transition is described by a complex-valued scalar order parameter $\psi(x,t) := \psi_r(x,t) + i\psi_i(x,t)$ with $|\psi(x,t)|^2$ representing the local density of superconducting electrons, where $\psi_r(x,t)$ and $\psi_i(x,t)$ denote the real part and imaginary part of $\psi(x,t)$ respectively and $i = \sqrt{-1}$ denotes the imaginary unit. The notation $\vec{A}(x,t)$ represents the real-valued magnetic vector potential such that $\nabla \times \vec{A} = \vec{B}(\vec{r},t)$ is the induced magnetic field inside the material. Additionally, the total strain inside the material is given by $\epsilon(x,t) = \frac{1}{2}(\nabla\vec{u} + (\nabla\vec{u})^T))$ which is the symmetric gradient of the displacement $\vec{u}$. The advantage of using a phase-field model to solve the system is that it empowers