

A Third Order Bound-Preserving Nodal Discontinuous Galerkin Method for Miscible Displacements in Porous Media

Xinyuan Liu¹, Tao Xiong^{2,*} and Yang Yang³

¹ School of Mathematics and Statistics, Heze University, Shandong, 274015, P.R. China.

² School of Mathematical Sciences, University of Science and Technology of China, Hefei, Anhui, 230026, P.R. China.

³ Department of Mathematical Sciences, Michigan Technological University, Houghton MI, 49931, USA.

Received 16 February 2023; Accepted (in revised version) 4 December 2023

Abstract. In this paper, we develop a provable third order bound-preserving (BP) nodal discontinuous Galerkin (DG) method for compressible miscible displacements. We consider the problem with a multi-component fluid mixture and physically the volumetric concentration of each component, $c_j (j=1, \dots, N)$, is between 0 and 1. The main idea is to apply a positivity-preserving (PP) method to all c_j 's, while enforce $\sum_j c_j = 1$ simultaneously. First, we treat the time derivative of the pressure as a source and choose suitable "consistent" numerical fluxes in the pressure and concentration equations to construct a nodal interior penalty DG (IPDG) method to enforce $\sum_j c_j = 1$. For PP, we represent the cell average of c_j as a weighted summation of Gaussian quadrature point values, and transform which to some other specially chosen point values. We prove that by taking appropriate parameters in the nodal IPDG method and a suitable time stability condition, the cell average can be kept positive, which further implies that the cell averages of all components are between 0 and 1. Finally, we apply a polynomial scaling limiter to obtain physically relevant numerical approximations without sacrificing accuracy. Numerical experiments are given to demonstrate desired accuracy, BP and good performances of our proposed approach.

AMS subject classifications: 65M60, 76T10, 35K40

Key words: Miscible displacements, bound-preserving, nodal discontinuous Galerkin method, multi-component fluid.

*Corresponding author. Email addresses: xinyuanliu@stu.xmu.edu.cn (X. Liu), taoxiong@ustc.edu.cn (T. Xiong), yyang7@mtu.edu (Y. Yang)

1 Introduction

In this paper we will construct a third order bound-preserving (BP) nodal discontinuous Galerkin (DG) scheme for N -component compressible miscible displacements in porous media. The classical model given by [17] over the computational domain $\Omega = [a, b] \times [c, d]$ reads

$$\mathbf{u} = -\frac{\kappa(x, y)}{\mu(\mathbf{c})} \nabla p, \quad (x, y) \in \Omega, \quad (1.1a)$$

$$d(\mathbf{c}) \frac{\partial p}{\partial t} + \nabla \cdot \mathbf{u} = q, \quad (x, y) \in \Omega, \quad 0 < t \leq T, \quad (1.1b)$$

$$\phi \frac{\partial c_j}{\partial t} + \nabla \cdot (\mathbf{u} c_j) - \nabla \cdot (\mathbf{D} \nabla c_j) = \tilde{c}_j q - \phi c_j z_j \frac{\partial p}{\partial t}, \quad (x, y) \in \Omega, \quad 0 < t \leq T, \quad j = 1, \dots, N-1. \quad (1.1c)$$

In (1.1), p , \mathbf{u} and \mathbf{c} are the unknown variables. p is the pressure in a fluid mixture, $\mathbf{u} = (u_1, u_2)^T$ represents the Darcy velocity of the mixture (volume flowing across a unit across-section per unit time), and $\mathbf{c} = (c_1, \dots, c_j, \dots, c_N)^T$, where c_j denotes the (volumetric) concentration of the j th component of a fluid mixture, which should be between 0 and 1. However, c_j does not satisfy a maximum-principle, i.e. if $c_j(x, y, t = 0) \in [m, M]$, $c_j(x, y, t > 0)$ may be beyond this range unless $m = 0$ and $M = 1$. $\kappa(x, y)$ is the permeability of the rock and $\mu(\mathbf{c})$ is the viscosity of the fluid. $\phi = \phi(x, y)$ is the porosity of the rock, q is the external volumetric flow rate, and \tilde{c}_j is the concentration of the j th component in the external flow. \tilde{c}_j must be specified at points at which injection (i.e., $q > 0$) takes place, and \tilde{c}_j is assumed to be equal to c_j at production points (i.e., $q < 0$). Moreover, $\sum_{j=1}^N \tilde{c}_j = 1$. The diffusion coefficient \mathbf{D} combines the effects of molecular diffusion, which is rather small for field-scale problems, and dispersion, which is velocity-dependent. It has the form

$$\mathbf{D} = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} = \phi(x, y) (d_{\text{mol}} \mathbf{I} + d_{\text{long}} |\mathbf{u}| \mathbf{E} + d_{\text{tran}} |\mathbf{u}| \mathbf{E}^\perp), \quad (1.2)$$

where \mathbf{E} , a 2×2 matrix, represents the orthogonal projection along the velocity vector given as

$$\mathbf{E} = (e_{ij}(\mathbf{u})) = \begin{pmatrix} u_i u_j \\ |\mathbf{u}|^2 \end{pmatrix}, \quad i = 1, 2, \quad j = 1, 2,$$

and $\mathbf{E}^\perp = \mathbf{I} - \mathbf{E}$ is the orthogonal complement. The diffusion coefficient d_{long} measures the dispersion in the direction of the flow and d_{tran} shows the transverse to the flow. In this paper, it is assumed that \mathbf{D} satisfies $D_{11} > 0$, $D_{22} > 0$ and $D_{11}D_{22} - D_{12}D_{21} > 0$. Moreover, the pressure is uniquely determined up to a constant, thus we assume $\int_\Omega p \, dx dy = 0$ at $t = 0$. However, this assumption is not essential. Other coefficients can be stated as follows:

$$c_N = 1 - \sum_{j=1}^{N-1} c_j, \quad d(\mathbf{c}) = \phi \sum_{j=1}^N z_j c_j,$$