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Non-Iterative Parallel Schwarz Domain Decomposition Algorithms for a Two-Domain Parabolic Problem

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Abstract. Based on overlapping domain decomposition, two parallel Schwarz finite element algorithms are presented for solving a two-domain parabolic problem. Our algorithms are based on two ideas. One is the observation that the influence of the non-homogeneous boundary data diminishes exponentially towards the interior of the subdomains. Another is to apply proper partitions of unity to distribute the residuals of the discretized systems reasonably into many subdomains so as to avoid repeated correction in the overlapping parts. The resulting algorithms are fully parallel in each subdomain. At each time step, only one iteration step is required to reach the optimal order of accuracy. The convergence rate depending on the mesh parameters are analyzed. Finally, numerical results are reported, which verify the theoretical analysis.

AMS subject classifications: 65M55, 65M60, 65M12

Key words: Domain decomposition, parallel Schwarz method, parabolic problems, error estimate.

1 Introduction

Domain decomposition methods (DDMs) [24] are powerful iteration methods for solving large-scale elliptic equations and other stationary problems. By virtue of domain decomposition, a large or complicated domain can be decomposed into several small overlapping or non-overlapping subdomains, such that a stationary problem can be divided into a set of sub-problems, each defined on a subdomain and can be solved effectively. The earliest known overlapping domain decomposition method was the

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Schwarz alternating method [22], which was proposed in the nineteenth century. Although Schwarz's motivation was primarily theoretical, the iterative formulation underlying Schwarz's method can be applied to solve a wide range of elliptic problems. With the development of domain decomposition methods, many new ideas and techniques have been developed, such as subspace correction methods [31, 32], parallel multilevel precondition algorithms [5], multiplicative Schwarz methods [4, 10], additive Schwarz methods [6, 8, 9, 16–18], parallel weighted Schwarz methods [29]. Xu [31] observed that with the notation of space decomposition, a wide range of iterative algorithms including domain decomposition methods can be presented in a unified mathematical framework, in which each iteration step is divided into two steps: at the first step, residual equations are solved either successively or in parallel within subspaces, and then at the second step, local corrections are summed to produce a global solution. If the first step is carried out in parallel, then this method is called parallel subspace correction method (PSC), which is not always convergent. However, in the context of domain decomposition methods, the preconditioner obtained from PSC is known as the additive Schwarz method (AS) [6,16], which greatly improves the condition number of the original problem. In [8, 9], Cai proposed the restricted additive Schwarz preconditioners (RAS) and the restricted additive Schwarz preconditioners with harmonic overlaps (RASHO), which surpass the performance of the classical AS preconditioners. The RAS preconditioner and its variants are widely employed in literature. For example, cf. [20,21,23,35,37,38]. In [29], Lu introduced the parallel weighted Schwarz algorithm (PWS) and established the convergence result. Their approach is similar to PSC, while an arithmetic mean is utilized for the local corrections. These overlapping domain decomposition algorithms are effective for elliptic problems.

Recently, overlapping domain decomposition methods were applied to solve the linear system arising from a time discretization of a parabolic problem. At each time level, the resulting equation is equivalent to an elliptic problem depending on a time step parameter τ, which suggests that one can extend any domain decomposition algorithms to parabolic problems. Moreover, by taking advantage of the time stepping, it is possible to design domain decomposition algorithms with fewer iteration steps at each time level. In [7], Cai extended the additive Schwarz preconditioners to parabolic equations and proved that the coarse mesh space can be avoided provided that τ/H^2 is reasonable small. Herein H denotes the maximal diameter of the subdomains. Based on non-overlapping domain decomposition, Dawson [13,14] proposed the explicit/implicit algorithms, in which the boundary condition at intersection points are predicted by explicit schemes and thus a stability condition $\tau=\mathcal{O}(H^2)$ is required, which is less restrictive than the time step restriction of the fully explicit scheme. Blum [3] proposed the domain splitting algorithms, where the boundary values are predicted from an extrapolation procedure and the global solutions are obtained by merging the patchwise solutions on the non-overlapping parts. These two approaches are non-iterative. In [10], Cai proposed the multiplicative Schwarz algorithms for parabolic equations and studied the convergence rate. Tai [28] proposed the parallel weighted Schwarz algorithms for parabolic equations