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Integral Regularization PINNs for Evolution Equations

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Abstract. Evolution equations, including both ordinary differential equations (ODEs) and partial differential equations (PDEs), play a pivotal role in modeling dynamic systems. However, achieving accurate long-time integration for these equations remains a significant challenge. While physics-informed neural networks (PINNs) provide a mesh-free framework for solving PDEs, they often suffer from temporal error accumulation, which limits their effectiveness in capturing long-time behaviors. To alleviate this issue, we propose integral regularization PINNs (IR-PINNs), a novel approach that enhances temporal accuracy by incorporating an integral-based residual term into the loss function. This method divides the entire temporal interval into smaller subintervals and enforces integral constraints either within each subinterval or across intervals extending from the initial moment to the current one, thereby improving the resolution and correlation of temporal dynamics. Furthermore, IR-PINNs leverage adaptive sampling to dynamically refine the distribution of collocation points based on the evolving solution, ensuring higher accuracy in regions with sharp gradients or rapid variations. Numerical experiments on benchmark problems demonstrate that IR-PINNs outperform original PINNs and other state-of-the-art methods in capturing long-time behaviors, offering a robust and accurate solution for evolution equations.

AMS subject classifications: 65L04, 65M99, 68T07

Key words: Deep learning, evolution equation, long-time integration, adaptive sampling.

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1 Introduction

Evolution equations are fundamental in modeling a wide range of physical, biological, and engineering phenomena, spanning from fluid dynamics to material science [5]. Among these, evolution equations, characterized by their dependence on both temporal and spatial variables, play a crucial role in describing dynamic systems. However, achieving accurate and efficient solutions to these equations remains a significant challenge, particularly for problems requiring long-time integration or involving complex and high-dimensional domains.

In recent years, physics-informed neural networks (PINNs) [24] have emerged as a promising alternative for solving evolution equations. By embedding the governing equations into the loss function and leveraging the expressive power of neural networks, PINNs can approximate solutions without relying on predefined grids or explicit discretization schemes. Despite their versatility, challenges remain in applying PINNs to evolution equations [6,17,25,28,30]. One of the most pressing issues is the accumulation of temporal errors during long-time integration, prompting significant efforts to address this limitation. Several training strategies have been proposed to improve temporal accuracy, including sequential learning [20,31], causal training [14,22,27] and operator learning [18,26,29,32]. Additionally, hybrid strategies have been developed to combine classical numerical methods with deep learning techniques. These approaches either adapt neural networks to augment classical PDE solvers [2,8] or incorporate classical numerical methods to enhance the performance of PINNs [4,10,13].

For evolution equations, the solution at any given time is inherently dependent on its state at previous times, reflecting strong temporal correlation. However, original PINNs treat temporal collocation points in isolation, failing to explicitly account for these correlations. This limitation often leads to challenges in capturing long-time dynamics and results in temporal error accumulation. Inspired by the integral form of evolution equations, we introduce a regularization term into the training process, proposing a novel framework termed integral regularization PINNs (IR-PINNs). Our main contributions can be summarized as:

- We propose integral regularization PINNs (IR-PINNs) for evolution equations by dividing the entire temporal interval into smaller subintervals, reformulating the evolution equation into an integral form, and incorporating an integral-based residual term into the loss function. This approach enhances temporal accuracy by enforcing constraints over specific temporal subintervals, thereby improving the resolution and correlation of temporal dynamics.

- We extend IR-PINNs with an adaptive sampling strategy, which dynamically refines the distribution of spatial collocation points, ensuring higher accuracy in regions with sharp gradients or rapid variations.

- We conduct numerical experiments on benchmark problems, including both linear