

## United Filter for Jointly Estimating State and Parameters of Stochastic Dynamical Systems

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**Abstract.** This paper tackles the intricate task of jointly estimating state and parameters in data assimilation for stochastic dynamical systems that are affected by noise and observed only partially. While the concept of “optimal filtering” serves as the customary approach to estimate the state of the target dynamical system, traditional methods such as Kalman filters and particle filters encounter significant challenges when dealing with high-dimensional and nonlinear problems. When we also consider the scenario where the model parameters are unknown, the problem transforms into a joint state-parameter estimation problem. Presently, the leading-edge technique known as the Augmented Ensemble Kalman Filter (AugEnKF) addresses this issue by treating unknown parameters as additional state variables and employing the Ensemble Kalman Filter to estimate the augmented state-parameter vector. Despite its considerable progress, AugEnKF does exhibit certain limitations in terms of accuracy and stability. To address these challenges, we introduce an innovative approach, referred to as the United Filter. This method combines a remarkably stable and efficient ensemble score filter (EnSF) for state estimation with a precise direct filter dedicated to online parameter estimation. Utilizing the EnSF’s generative capabilities grounded in diffusion models, the United Filter iteratively fine-tunes both state and parameter estimates within a single temporal data assimilation step. Thanks to the robustness of the EnSF, the proposed United Filter method offers a promising solution for enhancing our understanding and modeling of dynamical systems, as demonstrated by results from numerical experiments.

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## 1 Introduction

State and parameter estimation present critical challenges in the field of data assimilation. The state estimation problem typically involves a noise-perturbed stochastic dynamical system, where the state may not be directly observable. Instead, partial noisy observations are used to obtain the optimal estimate for the state of the dynamics. The primary mathematical tool for addressing state estimation is known as “optimal filtering”. The goal of the optimal filtering problem is to construct an approximation for the conditional probability density function (PDF) of the target state conditioning on the observational data, referred to as the “filtering density”, and the optimal state estimation is formulated as the corresponding conditional expectation. Important methods for solving the optimal filtering problem include Bayesian filters such as Kalman-type filters [18, 24–26] and particle filters [1, 14, 15, 20, 27, 40, 48], along with other optimal filtering methods related to solving stochastic partial differential equations (SPDEs) [5, 8–10, 16, 19, 32, 52, 53]. Although each of these methods has uniquely contributed to the realm of optimal filtering, they all have major drawbacks that cannot be easily addressed. For example, Kalman filters, relying on the Gaussian assumption for state distributions, can handle relatively high-dimensional problems. But the Gaussian assumption also makes the Kalman filter (and its extended versions) vulnerable when dealing with the nonlinearity that arises in filtering problems [47]. The particle filter, which is also known as a sequential Monte Carlo approach, can alleviate the nonlinearity issue in optimal filtering problems with moderate dimension. However, due to limitations of the Monte Carlo method and challenges associated with high dimensional sampling in Bayesian inference, extending particle filters to solve high dimensional problems is a formidable task [43]. While the SPDE-related methods are mathematically solid and can provide stable state estimation performance over time, the higher computational cost usually accompanying with SPDE solvers makes the SPDE-related optimal filtering methods less popular in those applications involving extensive computations, such as weather forecasting and state estimation for power systems.

In most studies on state estimation, there is a common assumption that the dynamics of the state are explicitly given. However, practical situations often involve state models with unknown factors. This leads to the challenge of parameter estimation. Approaches for parameter estimation can be categorized into the point estimation approach, such like maximum likelihood estimation (MLE) [50] and maximum a posteriori estimate (MAP) [39], and the Bayesian approach. When integrating parameter estimation with state estimation through optimal filtering, a natural choice is to employ a recursive Bayesian inference approach for joint online parameter estimation and dynamic state estimation. The current state-of-the-art method for joint state-parameter estimation is the Augmented Ensemble Kalman Filter (AugEnKF) [31]. This method considers unknown model parameters as additional state variables and analyzes them alongside other state variables through the ensemble Kalman filter (EnKF). Apparently, the success of the AugEnKF heavily depends on the performance of the EnKF for state estimation since pa-