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Enhanced Second-Order Gauss-Seidel Projection Methods for the Landau-Lifshitz Equation

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Abstract. The dynamics of magnetization in ferromagnetic materials are modeled by the Landau-Lifshitz equation, which presents significant challenges due to its inherent nonlinearity and non-convex constraint. These complexities necessitate efficient numerical methods for micromagnetics simulations. The Gauss-Seidel Projection Method (GSPM), first introduced in 2001, is among the most efficient techniques currently available. However, existing GSPMs are limited to first-order accuracy. This paper introduces two novel second-order accurate GSPMs based on a combination of the biharmonic equation and the second-order backward differentiation formula, achieving computational complexity comparable to that of solving the scalar biharmonic equation implicitly. The first proposed method achieves unconditional stability through Gauss-Seidel updates, while the second method exhibits conditional stability with a Courant-Friedrichs-Lewy constant of 0.25. Through consistency analysis and numerical experiments, we demonstrate the efficacy and reliability of these methods. Notably, the first method displays unconditional stability in micromagnetics simulations, even when the stray field is updated only once per time step.

AMS subject classifications: 35K61, 65M06, 65Z05

Key words: Gauss-Seidel projection methods, Landau-Lifshitz equation, backward differentiation formula, second-order accuracy.

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1 Introduction

Ferromagnetic materials are ideal candidates for magnetic recording applications due to their intrinsic magnetic properties. This class of materials locally exhibits a net magnetic moment, known as magnetization, in the absence of external magnetic fields. The distribution of magnetization in the magnetic body gives rise to various static magnetic textures, such as the domain wall, magnetic vortex and skyrmions. When the magnetic field, spin-polarized current or temperature gradient is applied, the magnetic textures would be driven, collapsed and generated, where the dynamics of the magnetization is guided by the Landau-Lifshitz (LL) equation [7,8].

The LL equation consists of the gyromagnetic and damping terms. The gyromagnetic term models the precession of the magnetization and conserves the magnetic free energy of the system, whereas the damping term governs the dissipation of energy. Below Curie's temperature, the length of magnetization is a constant at a point-wise level, and this infers the non-convex constraint in the LL equation. In addition, both the gyromagnetic term and damping term are strongly nonlinear. These accumulated features pose challenges in designing efficient and stable numerical methods for simulating the LL equation in micromagnetics simulations.

Over the past several decades, various numerical approaches have been developed for the micromagnetics simulations, see e.g. review papers [4–6] and references therein. The explicit methods, such as Runge-Kutta methods [13, 14], are favored in the early days. Explicit methods commonly suffer conditional stability where a small temporal step size is allowed, which results in the low efficiency of micromagnetics simulations even when certain adaptive strategies are adopted. On the contrary, the implicit Crank-Nicolson [15, 16] method maintains the length of magnetization and the energy law simultaneously. Nevertheless, the solvability of the implicit methods is strictly constrained with a ratio between the mesh size and time step size, and the iteration solver is required in applications due to the nonlinearity. These issues can be addressed by using linearized implicit methods, also known as the semi-implicit methods, such as the semi-implicit Euler method [17], implicit-explicit approaches [18], the semi-implicit backward differentiation formula (BDF) methods [11,12] and the semi-implicit Crank-Nicolson method [19]. Semi-implicit methods are characterized by their unconditional stability, unique solvability, and high efficiency. When a normalized step is further applied, the length of magnetization is preserved, although the energy law cannot be strictly maintained. Conversely, in linearized implicit methods, the preservation of magnetization length is not guaranteed, which may lead to inaccurate solution in long-time simulations. In this regard, high-order accurate methods exhibit their advantages, and some recent progress has been made such as the predictor-corrector methods [20] and IMEX-RK method [21]. In practice, the high-order methods not only provide a more accurate numerical solution but also can release the constriction of time step size. An intriguing example is provided in [10], where the use of the second-order BDF (BDF2) scheme permits a larger time step for simulating the LL equation with Dzyaloshinskii-Moriya interaction.