Commun. Comput. Phys.
doi: 10.4208/cicp.OA-2023-0236

Vol. **39**, No. 3, pp. 884-918
March 2026

Towards the Efficient Calculation of Quantity of Interest from Steady Euler Equations II: A CNNs-Based Automatic Implementation

Jingfeng Wang[1] and Guanghui Hu[1,2,*]

[1] *State Key Laboratory of Internet of Things for Smart City and Department of Mathematics, University of Macau, Macao, China.*

[2] *Zhuhai UM Science & Technology Research Institute, Zhuhai, Guangdong Province, China.*

Communicated by Chi-Wang Shu

Received 4 September 2023; Accepted (in revised version) 21 January 2024

Abstract. In [J. Wang, G. Hu, arxiv: 2302.14262], a dual-consistent dual-weighted residual-based h-adaptive method has been proposed based on a Newton-GMG framework, toward the accurate calculation of a given quantity of interest from Euler equations. The performance of such a numerical method is satisfactory, i.e., the stable convergence of the quantity of interest can be observed. In this paper, we will focus on the efficiency issue to further develop this method. Three approaches are studied for addressing the efficiency issue, i.e., i). using convolutional neural networks as a solver for dual equations, ii). designing an automatic adjustment strategy for the tolerance in the h-adaptive process to conduct the local refinement and/or coarsening of mesh grids, and iii). introducing OpenMP, a shared memory parallelization technique, to accelerate the module such as the solution reconstruction in the method. The feasibility of each approach and numerical issues are discussed in depth, and significant acceleration from those approaches in simulations can be observed clearly from a number of numerical experiments. In convolutional neural networks, it is worth mentioning that the dual consistency plays an important role in guaranteeing the efficiency of the whole method and that unstructured meshes are employed in all simulations.

AMS subject classifications: 65M50, 65N08

Key words: DWR-based h-adaptivity, dual consistency, convolutional neural networks, automatic adjustment of tolerance, Newton-GMG method for Euler equations.

*Corresponding author. *Email addresses:* garyhu@um.edu.mo (G. Hu), yc07410@um.edu.mo (J. Wang)

1 Introduction

In [1], a dual-consistent dual-weighted residual(DWR)-based adaptive mesh method has been constructed, from which the smooth convergence of quantity of interest can be observed. This technique is particularly advantageous in applications such as the optimal design of vehicle shapes where the quantity of interest, e.g., the lift-to-drag ratio, needs an accurate evaluation.

It is noted that the efficiency of a numerical method is a key feature towards the practical applications, as in a typical problem of optimal design of the vehicle shape, governing partial differential equations (PDEs) need to be solved numerous times, depending on factors such as the number of design parameters, mesh resolution, optimization method [2]. In our previous work, different modules like reconstruction [3–5], NURBS enhancement [6] and DWR-based refinement [7, 8] are integrated. As a consequence, in this paper, efforts are devoted to further improving the efficiency of the numerical method proposed in [1], to make the method and related library AFVM4CFD a competitive one in the market.

For such a purpose, three approaches will be studied in depth in this paper, including i). using convolutional neural networks (CNNs) to produce numerical solutions of the dual problem, ii). designing an automatic adjustment strategy for the tolerance in the h-adaptive process, and iii). introducing OpenMP to parallelize modules such as the solution reconstruction and dual solver in the method.

It is noted that Galerkin orthogonality between numerical solutions and related error is the reason why Hartmann [9] suggested solving dual equations in a larger space. In [1], the refinement of mesh grids is employed for such a purpose. However, this approach is not a good choice from the efficiency point of view [10]. Although there have been other choices such as raising the order of approximate polynomials [11], high order interpolation of numerical solutions of dual equations [12], it should be pointed out that, i). solving dual equations using a classical solver involved in all aforementioned approaches, in which the time-consuming modules such as the construction of the finite-dimensional space, solving the system of linear equations, etc., need to be implemented, and ii). numerical solutions of dual equations are adopted as a weight for the mesh adaptation in the current framework. Out of the motivation to develop a high-quality mesh, the trained neural network model can fulfill this requirement while saving time significantly.

CNNs, based on the above two observations, become an ideal choice for the solver of dual equations, due to their ability to fastly generate acceptable numerical solutions of dual equations. In the recent past, neural networks have been successfully applied to implement the DWR-based h-adaptation method in computational fluid dynamics (CFD). For instance, in [13, 14], an encoder-decoder algorithm is developed to predict the indicators generated from dual equations, and in [15], primal solutions and dual solutions are both obtained from deep neural networks. Similarly, a data-driven goal-oriented mesh adaptation approach is developed to generate the error indicators in [16]. The recent breakthrough in convolutional neural networks makes training in high-dimensional