

Derivation of Asymptotic Limiting Equations for Multigroup Anisotropic-Scattering Transport Processes in Optically Thick Regimes

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Abstract. Asymptotic limit methods play a crucial role in deriving macroscopic fluid equations from equations governing mesoscopic particle transport mechanics, constituting a significant research area within kinetic theory. In this paper, we present asymptotic expansions applied to the multigroup linear transport equation with a general anisotropic scattering kernel. We derive the diffusion limit for such transport equations under three different scalings. For the conventional scaling, the standard multigroup diffusion equations are derived as an asymptotic approximation to the multigroup transport equations. For the other two scalings where the scattering effect between groups is strong, new group-collapsed (monoenergetic) diffusion approximations with different coefficients depending on the form of the scaling are derived. Numerical experiments validate the prediction of the theoretical models. This paper provides a deeper understanding of the diffusion physics for the multigroup anisotropic particle transport in the optically thick regime.

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1 Introduction

Particle transport processes are prevalent in aerodynamics, nuclear physics, and chemical engineering. It is also the essential energy transfer process in high-energy-density engineering [4, 8, 13]. An important parameter describing particle transport behavior is

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the Knudsen number, defined as the ratio between the mean free path of particle transport and the macroscopic length scale. The Boltzmann equation describes the microscopic behavior of particle transport from a statistical viewpoint [1] and is valid for all Knudsen numbers. However, despite its detailed microscopic description of particle behavior, solving the Boltzmann equation numerically can be computationally demanding. Therefore, in practical applications, approximate macroscopic models, often in the form of diffusion equations, are commonly employed. Establishing a mathematically rigorous link between transport theory and approximate diffusion theory, as outlined in Hilbert's sixth problem, stands as one of the foremost objectives of kinetic theory. Besides its theoretical significance, deriving macroscopic transport equations from the Boltzmann equation for small Knudsen numbers contributes to the development of efficient numerical methods for the simulation of the rarefied gas dynamics and the transport of neutral particles [2,7,15].

For the transport process of neutral particles, Larsen et al. developed the asymptotic low-order diffusion models in the optically thick regime with isotropic scattering [5,10,11]. Pomraning considered linear anisotropic scattering in [12]. In [6], the asymptotic analysis is extended to an anisotropic scattering transport process with the scattering kernel represented by an arbitrary-order Legendre polynomial. Most of the published asymptotic analyses of transport problems deal with monoenergetic (one-group) problems. [9] presents an asymptotic theory for multigroup, three-dimensional, isotropically scattering, homogeneous-medium neutron transport k-eigenvalue problems. However, the asymptotic analysis of transport processes characterized by both multigroup and anisotropic scattering is still open.

The primary objective of this paper is to derive the diffusion limit for different variations of the collision and time-scaled multigroup time-dependent transport equation with a general anisotropic scattering kernel. We are specifically interested in the diffusion equations that asymptotically approximate the multigroup transport equations under the three different scalings investigated in [9]: the conventional scaling where in-group scattering dominates, the weak-equilibrium diffusion scaling where out-group scattering is increased but still of smaller magnitude than the in-group scattering, and the strong-equilibrium diffusion scaling where out-group scattering is of the same order-of-magnitude as the in-group scattering. Our main conclusions are that for the conventional scaling, the traditional multigroup diffusion equation is the asymptotic limit of the multigroup anisotropic transport process, where the diffusion coefficients rely on the first-order moments of the scattering kernel. For the two alternative scalings discussed in [9], where out-group scattering is strong, the anisotropic transport equation limits to different forms of the diffusion equation, the coefficients of which depend on the first-order moments of the scattering kernel, but not on their higher-order moments. For the special case where the scattering is isotropic, our analysis is consistent with the conclusion in [9], that the two scalings yield the same 'group-collapsed' diffusion limits. The mathematical derivation and numerical experiments are presented in the following sections.