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Least-Squares Recovery of Dual Variables in ROMs for Saddle-Point Problems

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Abstract. In this work, we introduce a methodology for recovering the reduced Lagrange multiplier in Reduced Order Models (ROMs) of general saddle-point problems. Specifically, the multiplier is determined as the least-squares minimizer of the residual of the reduced primal variable in a dual norm. We prove that this procedure yields a unique solution under the condition that the full-order primal and dual spaces satisfy the inf-sup stability condition. This is an extension to general saddle-point problems of the method recently introduced in [6] to recover the reduced pressure in ROMs of incompressible flows. We further show that the proposed approach is equivalent to solving the reduced mixed problem with an enriched reduced primal basis, augmented by the supremizers of the reduced multiplier. We establish optimal error estimates for the reduced multiplier applicable to general saddle-point problems. Numerical experiments for heat diffusion in composite media validate our theoretical findings and demonstrate the suitability of the method to solve engineering problems.

AMS subject classifications: 65N15

Key words: Reduced order models (ROM), saddle point problems, dual variable recovery.

1 Introduction

Saddle-point problems are a cornerstone of modern applied mathematics, underpinning a wide variety of theoretical and computational frameworks across multiple disciplines. These problems naturally arise in systems where equilibrium conditions must satisfy

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competing constraints or where primal-dual relationships are central to the formulation. Their significance stems not only from their theoretical elegance but also from their widespread applicability to real-world phenomena.

In fluid dynamics, saddle-point structures are intrinsic to modeling incompressible flows, as seen in the Stokes and Navier-Stokes equations [11, 46]. Here, the coupling between velocity and pressure fields is governed by the incompressibility constraint, a classical saddle-point condition. Similarly, in structural mechanics, mixed finite element methods for elasticity and contact problems rely on saddle-point formulations to ensure the correct interaction between stress and displacement variables [11, 13].

Beyond physics, saddle-point problems play a critical role in optimization, particularly in constrained and minimax problems. In machine learning, many problems naturally lead to saddle-point formulations. A prominent example is adversarial training in generative adversarial networks (GANs), where a min-max optimization problem defines a saddle-point structure between a generator and a discriminator [22]. Similarly, support vector machines (SVMs) can be expressed through saddle-point formulations using Lagrangian duality [18]. More generally, constrained empirical risk minimization problems, particularly in robust optimization and distributionally robust learning, often take the form of min-max or max-min problems [1, 2]. These saddle-point problems require specialized algorithms, such as primal-dual or extragradient methods, to ensure stability and convergence in high-dimensional parameter spaces. Likewise, in economics and game theory, saddle-point structures encapsulate equilibrium conditions, providing a mathematical framework for understanding agent interactions and optimization hierarchies [5, 48].

In computational electromagnetics, saddle-point formulations enable efficient simulation of Maxwell's equations and magnetostatic problems, where primal and dual variables such as electric and magnetic fields are naturally intertwined [12, 34]. Moreover, mathematical physics and optimal control theory frequently rely on saddle-point problems to couple states and controls, offering robust solutions to complex systems governed by partial differential equations [21].

The ubiquity and importance of saddle-point problems demand robust and efficient numerical techniques to solve them, particularly in reduced order settings where computational efficiency is critical.

Reduced Order Modelling (ROM) provides a broad range of both intrusive and non-intrusive techniques that provide fast solutions of parametric problems in fluid and solid mechanics, electromagnetism, optimisation and control, among other areas (see, for instance, the comprehensive reference [9]). ROM is an artificial intelligence procedure in which the off-line learning step uses the information of the so-called "snapshots" (solutions of the parametric problem afforded for selected values of the parameters) to construct a low-dimension space or variety of solutions. In the on-line step the intrusive ROMs solve low-dimensional discretisations of the problem afforded, while the non-intrusive ROMs use generalised interpolatory techniques.