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# High-Order Structure-Preserving Spectral Difference Methods for Hyperbolic Conservation Laws

Yupeng Ren[1,*] and Zhimin Zhang[2]

[1] *School of Mathematics and Information Science & Center for Applied Mathematics of Guangxi (Guangxi University), Guangxi University, Nanning, 530004, China.*
[2] *Department of Mathematics, Wayne State University, Detroit, MI 48202, USA.*

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**Abstract.** This paper presents a study focused on solving hyperbolic conservation laws using arbitrary-order spectral difference (SD) methods. The study is structured around several crucial aspects. Firstly, in order to ensure the maximum principle for scalar equations and positivity preservation for Euler systems, we adopt the concept of flux limiters. This adoption leads to the development of the structure-preserving SD scheme with a flux limiter (SDFL), which is proven to preserve the original high-order accuracy. However, the SDFL scheme with lower order might lack conservational properties, despite its strong performance in short-term simulations. Consequently, we have developed a specific variant of the SDFL scheme with conservational properties, referred to as the CSDFL scheme. Secondly, we introduce a modified WENO-ZQ (MWENO-ZQ) reconstruction to suppress spurious oscillations when simulating problems with strong discontinuities. Finally, we conduct extensive numerical experiments to validate the effectiveness of the proposed high-order SD (SDFL,CSDFL) methods with MWENO-ZQ reconstruction. The results demonstrate the robustness and efficiency of these techniques in solving problems involving strong discontinuities, low pressure, and low density.

**AMS subject classifications**: 76M22, 65M06, 35L65
**Key words**: Spectral difference method, conservation law, maximum principle, positivity-preserving.

## 1 Introduction

In this paper, we consider the following conservation laws

$$\mathbf{U}_t+\mathbf{F}(\mathbf{U})_{\mathbf{x}}=0, \quad \mathbf{x}\in\Omega, \tag{1.1}$$

*Corresponding author. *Email addresses:* `ypren@gxu.edu.cn` (Y. Ren), `ag7761@wayne.edu` (Z. Zhang)

 

where $\Omega \in \mathbb{R}^d$ is the computational domain and $\mathbf{U}$ can be either scalar or vector, depending on the problem. The function $\mathbf{F}$ is referred to as the flux function. For nonlinear flux functions like $\mathbf{F}$, even a sufficiently smooth initial value can lead to discontinuous solutions over time. Therefore, accurately capturing the shock is one of the most significant challenges in the numerical simulation of hyperbolic systems.

As computers and advancements in science and technology continue to progress, numerous higher-order numerical methods for hyperbolic conservation laws have been developed. Presently, the most widely used schemes include the WENO schemes [9] in the finite difference (FD) or finite volume (FV) framework, discontinuous Galerkin (DG) [18] in the finite element framework, spectral volume (SV) methods [22, 29], and spectral difference (SD) methods [11, 13]. The SD method occupies an intermediate position between the DG and FD frameworks. Its solution space resembles DG, with polynomials defined at certain Gaussian points (e.g., Chebyshev Gaussian points). The method originated from the "conservative Chebyshev multi-domain method with staggered lattice" proposed in [10]. However, the general formulation of this method was initially introduced by Liu in [13], specifically for conservation laws on unstructured grids. Subsequently, the method found applications in solving Euler equations and Navier-Stokes (NS) equations [20, 23]. Compared to the DG method, the SD method offers the advantage of not requiring volume or surface integration. In contrast to the FD method, the SD method possesses inherent compactness, simplifying boundary treatment and making it a more straightforward choice for implementation in parallel computing environments.

As previously mentioned, effectively capturing discontinuities has emerged as one of the challenges in numerical methods for hyperbolic conservation laws. In [13], the authors employed a TVB limiter for SD method to address the aforementioned challenge, originally designed for the SV method in [22]. However, it is important to note that the TVB limiter exhibits only second-order accuracy, which can potentially impact the accuracy of the higher-order method itself. Moreover, the parameters of the TVB limiter are problem-dependent, making it challenging to employ in practical situations.

Recently, WENO methods have been extensively utilized as limiters for DG methods in [16, 34]. Researchers also show a strong preference for the method of adding artificial viscosity, as evidenced in [14, 15]. However, there still exits user-defined parameters that might be influenced by the grid or the specific problem being addressed. In [11], the authors presented a nonlinear interpolation approach for the SD method, specifically designed to tackle conservation laws involving discontinuities. This nonlinear interpolation method is based on the multi-resolution WENO reconstruction initially proposed by [35]. However, the authors employed the WENO reconstruction for all reference cells, leading to a substantial increase in computational cost. This issue warrants further attention and consideration.

Another challenge in numerical simulation is that the numerical solution must be an entropy solution that adheres to the maximum principle. This ensures the physical relevance and stability of the solution, as well as the preservation of crucial properties in the equation under consideration. In simple terms, assuming that the initial value