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Steady-State Simulation of Euler Equations by the Discontinuous Galerkin Method with the Jump Filter

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Abstract. In this study, we introduce a jump filter for discontinuous Galerkin (DG) methods on both rectangular and triangular meshes, specifically designed for steady-state solutions of the Euler equations. Traditional limiters have occasionally been observed to produce slight post-shock oscillations, but the integration of the jump filter effectively reduces these fluctuations. This improvement allows the residual to stabilize at a level indistinguishable from machine zero in steady-state simulations. The jump filter operates by applying distinct damping factors to the different moments of the solution, based on the jump information at cell interfaces. This strategy creates transition zones from smooth regions to shock waves, which is critical for achieving steady-state solutions with residuals approaching the threshold of machine zero. The jump filter on unstructured meshes using Dubiner polynomials associated with the Sturm-Liouville operator is also investigated. Furthermore, the DG method with the jump filter retains many of the key advantages of the classical DG approach, including compactness, conservation, stability, and high-order accuracy. Extensive numerical experiments, conducted on both rectangular and triangular meshes, confirm the effectiveness of the jump filter in not only suppressing numerical oscillations but also significantly driving the residual towards machine zero.

AMS subject classifications: 65M60, 35L65, 76M10
Key words: Euler equations, discontinuous Galerkin method, local viscosity, jump filter, steady-state convergence, rectangular and triangular meshes.

1 Introduction

Steady states are prevalent in aerodynamics, where non-viscous flow fields are governed by the steady Euler equations:

$$\nabla\cdot \boldsymbol{F}(\boldsymbol{u}(\boldsymbol{x}))=0,\quad \boldsymbol{x}\in\Omega, \tag{1.1}$$

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where $\Omega \subset \mathbb{R}^d$ is an open polygonal domain with appropriate boundary conditions specified on $\partial\Omega$. Eq. (1.1) represents a system of $\mathcal{N} \in \mathbb{N}$ equations. The flux $\boldsymbol{F}(\boldsymbol{u}) = [\boldsymbol{F}_1(\boldsymbol{u}),\cdots,\boldsymbol{F}_d(\boldsymbol{u})]$ is a $\mathcal{N} \times d$ matrix, where the functions $\boldsymbol{F}_j$: $\mathbb{R}^{\mathcal{N}} \to \mathbb{R}^{\mathcal{N}}$ for $j=1,\cdots,d$, are given. Additionally,

$$\nabla \cdot \boldsymbol{F}(\boldsymbol{u}) = \sum_{j=1}^{d} \frac{\partial}{\partial x_j} \boldsymbol{F}_j(\boldsymbol{u}).$$

This paper focuses on the two-dimensional steady-state Euler equations, where $\mathcal{N}=4$ and $d=2$. The vector of conserved variables is defined as $\boldsymbol{u} = (\rho,\rho u,\rho v,E)^T$, where ρ is the density, $(u,v)^T$ is the velocity vector, and E is the total energy. The nonlinear flux function is given by $\boldsymbol{F}(\boldsymbol{u}) = (\boldsymbol{f}(\boldsymbol{u}),\boldsymbol{g}(\boldsymbol{u}))$, where $\boldsymbol{f}(\boldsymbol{u}) = (\rho u,\rho u^2+p,\rho uv,u(E+p))^T$ and $\boldsymbol{g}(\boldsymbol{u}) = (\rho v,\rho uv,\rho v^2+p,v(E+p))^T$. Here, p denotes the pressure, and γ is the specific heat ratio for air. To obtain the steady-state solutions of (1.1), we discretize the equations using the high-order discontinuous Galerkin (DG) method coupled with the pseudo-time marching method. When the residual is sufficiently small, the steady-state solutions are obtained.

The primary challenge in computing the steady-state solutions arises from the potential presence of pronounced discontinuities. Physical variables can experience sudden and significant changes, particularly in scenarios involving strong shocks or contact discontinuities. These challenges not only impede steady-state convergence but also often lead to unphysical oscillations near these strong discontinuities. Over the past three decades, various high-resolution schemes such as essentially non-oscillatory (ENO) scheme [9], weighted ENO (WENO) scheme [13], subcell finite volume shock capturing methods [22,29], arbitrary derivative (ADER) in space and time methods [6,30], targeted ENO (TENO) scheme [19] and DG methods have been developed. These approaches have proven highly effective in handling unsteady problems, especially by mitigating numerical oscillations induced by shock waves. For a comprehensive review of these high-resolution schemes and further insights, readers are referred to [28] and the associated references. However, applying these high-order numerical methods to steady-state problems has presented certain difficulties. In the early 1980s, Jameson et al. [11,12] proposed a method combining finite volume discretization, third-order dissipative terms, and a Runge-Kutta time-stepping scheme to effectively solve the Euler equations in arbitrary geometric domains. They successfully used this approach to determine steady transonic flow past an airfoil using an O-type mesh. Despite its effectiveness, this method has a notable drawback: the need to fine-tune parameters within the artificial viscosity for optimal performance. Yee et al. [40] addressed this issue by introducing an implicit, unconditionally stable high-resolution total variation diminishing (TVD) scheme, which they applied to compute steady-state solutions for compressible inviscid equations.

The DG method, initially crafted for neutron transport [26], was later expanded by the pioneering work of Cockburn et al. [1–3] to solve nonlinear hyperbolic conservation laws, including the time-dependent compressible Euler equations, using explicit Runge-Kutta time discretizations [27] combined with DG discretization in space. To prevent spurious